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Lie-Theoretic of Some Generating Relations of Hypergeometric Functions

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Abstract: *In this paper, we derived several generating relations involving the hypergeometric functions* $_2F_1(a,b;c;x)$ *by the group theoretical method known as Wisner's method. We have considered a three-parameter Lie group by giving a suitable interpretation to the numerator parameter* a *of the hypergeometric functions and obtained some known as well as some new generating relation for hypergeometric functions. Some specific cases of these relationships are also explored.*

1. INTRODUCTION

Marius Sophus Lie (1842-1899) [1] attempts to use the power of the tool called group theory to solve, or at least simplify ordinary differential equations. Earlier in the nineteenth century, ´Evariste Galois (1811-1832) had used group theory to solve algebraic (polynomial) equations that were quadratic, cubic, and quartic. Based on an analogy, Lie initiated his work. If finite groups were required to determine the solvability of finite degree equations, then the treatment of ordinary and partial differential equations would likely involve infinite groups.

We refer to the theoretical concepts of Lie [2] in this paper and derive the generation of function relationships involving hypergeometric function series $_2F_1(a,b;c;x)$. The theory of generating function relationships has been developed in different directions because of the important role that hypergeometric functions play in physics and applied mathematics problems, and has found wide applications in various branches of science and technology. Louis Weisner in [3] introduced Lie operators and then obtained generating function relations for generalized hypergeometric functions corresponding to increasing and decreasing of numerator parameter a. Manocha and Jain [4], Agrawal, and Jain [5] have also applied it to obtain generating function relations by variation of parameter a. This chapter is an attempt to exhibit the group-theoretic significance of generating function relations for hypergeometric functions. Weisner's group-theoretic method is utilized to obtain generating function relations of hypergeometric functions with the introduction of raising operators.

We begin by considering the following ordinary differential equation which satisfied by the hypergeometric functions [6] $y =_2 F_1(a,b;c;x)$:

$$x(1-x)\frac{d^2y}{dx^2}+[c-(a+b+1)x]\frac{dy}{dx}-aby=0 \qquad (1.1)$$

2. GROUP-THEORETIC DISCUSSION

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, a by $y\frac{\partial}{\partial y}$ and y by $u(x,y)$ in (1.1), we get the following partial differential equation:

$$x(1-x)\frac{\partial^2 u}{\partial x^2}+[c-(b+1)x]\frac{\partial u}{\partial x}-xy\frac{\partial^2 u}{\partial x\partial y}-by\frac{\partial u}{\partial y}=0. \qquad (2.1)$$

Thus $u(x,y)=y^a{}_2F_1(a,b;c;x)$ is a solution of (2.1), since $_2F_1(a,b;c;x)$ is a solution of (1.1).

We now seek linearly independent differential operators $J^3, J^-$and J^+ such that

$$\begin{cases} J^3\left(y^a{}_2F_1(a,b;c;x)\right)=a_n y^a{}_2F_1(a,b;c;x);\\ J^-(y^a{}_2F_1(a,b;c;x))=b_n y^{a-1}{}_2F_1(a+1,b;c;x);\\ J^+(y^a{}_2F_1(a,b;c;x))=c_n y^{a+1}{}_2F_1(a+1,b;c;x), \end{cases} \qquad (2.2)$$

where a_n, b_n and c_n are expressions in n which are independent of x and y. This necessitates the bringing into use of the following recurrence relations.

Theorem: The hypergeometric functions ${}_2F_1(a, b; c; x)$ satisfies the following recurrence relations

(i) $$[x(1-x)\frac{d}{dx} - bx + c - a]{}_2F_1(a, b; c; x) = (c-a){}_2F_1(a-1, b; c; x)$$

(ii) $$[x\frac{d}{dx} + a]{}_2F_1(a, b; c; x) = a{}_2F_1(a+1, b; c; x)$$

Proof:

(i) $$[x(1-x)\frac{d}{dx} - bx + c - a]{}_2F_1(a, b; c; x) = x(1-x)\frac{ab}{c}\ {}_2F_1(a+1, b+1; c+1; x) - (bx - c + a){}_2F_1(a, b; c; x)$$

$$= x(1-x)\frac{ab}{c}[1 + \frac{(a+1)(b+1)}{c+1}x + \ldots + \frac{(a+1)_n(b+1)_n}{(c+1)_n}\frac{x^n}{n!} + \ldots]$$

$$-(bx)[1 + \frac{ab}{c}x + \ldots + \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} + \ldots]$$

$$+(c-a)[1 + \frac{ab}{c}x + \ldots + \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} + \ldots]$$

Coffectionant of x^n

$$= \frac{(a-1)_n(b)_n}{(c)_n\ \ n!}[\frac{n(a+n-1)}{(a-1)} - \frac{n(n-1)(c+n-1)}{(b+n-1)(a-1)} - b\frac{n(c+n-1)}{(b+n-1)(a-1)} + (c-a)\frac{(a+n-1)}{(a-1)}]$$

$$= \frac{(a-1)_n(b)_n}{(c)_n\ \ n!}[\frac{(c-a+n)(a+n-1) - n(c+n-1)}{(a-1)}]$$

$$= \frac{(a-1)_n(b)_n}{(c)_n\ \ n!}[\frac{(a-1)(c+n-a) + nc + n^2 - an - nc - n^2 + n}{(a-1)}]$$

$$= \frac{(a-1)_n(b)_n}{(c)_n\ \ n!}[\frac{(a-1)(c+n-a) - an + n}{(a-1)}]$$

$$= \frac{(a-1)_n(b)_n}{(c)_n\ \ n!}[\frac{(a-1)(c+n-a) - n(a-1)}{(a-1)}]$$

$$= \frac{(a-1)_n(b)_n}{(c)_n\ \ n!}[c - a].$$

Thus

$$[x(1-x)\frac{d}{dx} - bx + c - a]\ {}_2F_1(a, b; c; x) = (c-a)\ {}_2F_1(a-1, b; c; x).$$

$$(ii)[x\frac{d}{dx} + a]{}_2F_1(a, b; c; x) = x\frac{ab}{c}\ {}_2F_1(a+1, b+1; c+1; x) + a{}_2F_1(a, b; c; x)$$

$$= x\frac{ab}{c}\left[1 + \frac{(a+1)(b+1)}{c+1}x + \cdots + \frac{(a+1)_n(b+1)_n}{(c+1)_n}\frac{x^n}{n!} + \cdots\right] + a[1 + \frac{ab}{c}x + \ldots + \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} + \ldots].$$

Coffectionant of x^n

$$= \frac{(a+1)_n(b)_n}{(c)_n\ \ n!}[\frac{na}{(a+n)} + \frac{a^2}{(a+n)}]$$

$$= \frac{(a+1)_n(b)_n}{(c)_n\ \ n!}[\frac{a(a+n)}{(a+n)}]$$

$$=\frac{(a+1)_n(b)_n}{(c)_n \quad n!}[a].$$

Thus

$$[x\frac{d}{dx}+a]_2F_1(a,b;c;x)=a \quad . \quad {}_2F_1(a+1,b;c;x).$$

With the aid of the above relations and (2.2), we get the following operators

$$\begin{cases} J^3 = y\frac{\partial}{\partial y}-\frac{1}{2}c; \\ J^- = y^{-1}[x(1-x)\frac{\partial}{\partial x}-bx-y\frac{\partial}{\partial y}+c]; \\ J^+ = y[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}]; \end{cases} \tag{2.3}$$

such that

$$J^3(y^a{}_2F_1(a,b;c;x))=(a-\frac{1}{2}c)y^a{}_2F_1(a,b;c;x);$$

$$J^-(y^a{}_2F_1(a,b;c;x))=(c-a)y^{a-1}{}_2F_1(a-1,b;c;x);$$

$$J^+(y^a{}_2F_1(a,b;c;x))=ay^{a+1}{}_2F_1(a+1,b;c;x).$$

Now we have the following commutator relations:

$$[J^3,J^\pm]=\pm J^\pm, \qquad [J^+,J^-]=2J^3, \tag{2.4}$$

where $[A,B]u=(AB-BA)u$.

The above commutator relations (2.4) show the set of operators J^3, J^- and J^+ generates a Lie algebra which is isomorphic to $sl(2)$.

Now we can write the differential operator as follow

$$L \equiv J^+J^- + (J^3+\frac{c}{2}-1)(J^3-\frac{c}{2}) \tag{2.5}$$

from which it follows that L commutes with each operator

$$[J^3,L]=[J^-,L]=[J^+,L]=0. \tag{2.6}$$

3. Extended Forms of the Groups Generated by J-Operators

To find the extended form of the group generated by J^3, we must solve the following differential equations

$$\frac{\partial y(a')}{\partial a'}=y(a') \tag{3.1}$$

and

$$\frac{\partial v(a')}{\partial a'}=-\frac{c}{2}v(a') \tag{3.2}$$

where $y(0)=y$, and $v(0)=1$.

From (3.1) we get

$$\int \frac{\partial y(a')}{\partial a'}\frac{1}{y(a')}da'=\int da'$$

$\log y(a')=a'+k.$

We put $a'=0$ to find the constant k, so we get

$\log y=k$

Then

$\log y(a') = a' + \log y$

$y(a') = ye^{a'}$

and from (3.2) we get

$$\int \frac{\partial v(a')}{\partial a'} \frac{1}{v(a')} da' = \int -\frac{c}{2} da'$$

$\log v(a') = -\frac{c}{2}a' + k.$

We put $a' = 0$ to find the constant k, so we get

$k = 0.$

Then

$\log v(a') = -\frac{c}{2}a'.$

$v(a') = e^{-\frac{c}{2}a'}.$

Thus

$$e^{a'J^3}u(x,y) = e^{-\frac{c}{2}a'}u(x, ye^{a'}). \tag{3.3}$$

Similarly, we get

$$e^{b'J^-}u(x,y) = (1-\tfrac{b'}{y})^{b-c}(1-\tfrac{b'}{y}(1-x))^{-b}u(\tfrac{xy}{y-b'(1-x)}, y-b'), \tag{3.4}$$

and

$$e^{c'J^+}u(x,y) = u(\tfrac{x}{1-c'y}, \tfrac{y}{1-c'y}). \tag{3.5}$$

Therefore

$$e^{a'J^3}e^{c'J^+}e^{b'J^-}u(x,y) = e^{-\frac{c}{2}a'}(1-\tfrac{b'}{y})^{b-c}(1-\tfrac{b'}{y}(1-x))^{-b} \tag{3.6}$$

$$\times \quad u(\frac{xy}{(y-b'(1-x))(1-c'(y-b'))}, \frac{(y-b')e^{a'}}{(1-c'(y-b'))})$$

4. Generating Function Relations

We will obtain generating function relations from the operator J^3 by considering the following two cases of the transformed function $\exp(c'J^+)\exp(b'J^-)\ {}_2F_1(a,b;c;x)$.

From (2.1) $u(x,y) = y^a{}_2F_1(a,b;c;x)$ is a solution of the system

$$\begin{cases} Lu = 0, \\ (J^3 - (a-\frac{c}{2}))u = 0. \end{cases} \tag{4.1}$$

Since L commutes with the operators, we have

$S(L)(y^a{}_2F_1(a,b;c;x)) = (L)S(y^a{}_2F_1(a,b;c;x)) = 0,$

where

$S = e^{a'J^3}e^{c'J^+}e^{b'J^-}.$

Therefore, the transformation $S(y^a{}_2F_1(a,b;c;x))$ is also annulled by L.

Now setting $a' = 0$ in (3.6), we get

$$e^{c'J^+}e^{b'J^-}u(x,y) = (1-\tfrac{b'}{y})^{b-c}(1-\tfrac{b'}{y}(1-x))^{-b} \tag{4.2}$$

$$\times \qquad u\left(\frac{xy}{(y-b'(1-x))(1-c'(y-b'))}, \frac{y-b'}{(1-c'(y-b'))}\right)$$

So, we obtain

$$e^{c'J^+}e^{b'J^-}[{}_2F_1(a,b;c;x)y^a] = (1-\tfrac{b'}{y})^{b-c}(1-\tfrac{b'}{y}(1-x))^{-b} \qquad (4.3)$$
$$\times {}_2F_1(a,b;c;\frac{xy}{(y-b'(1-x))(1-c'(y-b'))})(\frac{y-b'}{(1-c'(y-b'))})^a$$

So, we get the following special cases:

Case I : Let $b'=1,\ c'=0$, then (4.3) reduces to

$$e^{J^-}\left[y^a{}_2F_1(a,b;c;x)\right] = (1-\tfrac{1}{y})^{a+b-c}(1-\tfrac{1}{y}(1-x))^{-b} \quad {}_2F_1(a,b;c;\frac{x}{1-\frac{(1-x)}{y}})y^a \quad (4.4)$$

Now expanding this function, we get

$$e^{J^-}[y^a{}_2F_1(a,b;c;x)] = \sum_{m=0}^{\infty}\frac{(J^-)^m}{m!}[y^a{}_2F_1(a,b;c;x)]$$

$$= \sum_{m=0}^{\infty}\frac{(J^-)^{m-1}}{m!}[(c-a)y^{a-1}{}_2F_1(a-1,b;c;x)]$$

$$= \sum_{m=0}^{\infty}\frac{(J^-)^{m-m}}{m!}[(c-a)_m y^{a-m}{}_2F_1(a-m,b;c;x)]$$

Thus

$$e^{J^-}[y^a{}_2F_1(a,b;c;x)] = \sum_{m=0}^{\infty}\frac{(c-a)_m}{m!}[y^{a-m}{}_2F_1(a-m,b;c;x)]. \qquad (4.5)$$

Equating the two equations (4.4) and (4.5), we get

$$(1-\tfrac{1}{y})^{a+b-c}(1-\tfrac{1}{y}(1-x))^{-b}{}_2F_1(a,b;c;\frac{x}{1-\frac{(1-x)}{y}}) \qquad (4.6)$$

$$= \sum_{m=0}^{\infty}\frac{(c-a)_m}{m!}[y^{-m}{}_2F_1(a-m,b;c;x)]$$

Let $t=\frac{1}{y}$

$$(1-t)^{a+b-c}(1-t(1-x))^{-b}{}_2F_1(a,b;c;\frac{x}{(1-t(1-x))}) \qquad (4.7)$$

$$= \sum_{m=0}^{\infty}\frac{(c-a)_m}{m!}[{}_2F_1(a-m,b;c;x)t^m].$$

Case II : Let $b'=0,\ c'=1$, then (4.3) reduces to

$$e^{J^+}[y^a{}_2F_1(a,b;c;x)] = (\frac{y}{1-y})^a{}_2F_1(a,b;c;\frac{x}{1-y}). \quad (4.8)$$

Now expanding this function, we get

$$e^{J^+}[y^a{}_2F_1(a,b;c;x)] = \sum_{k=0}^{\infty}\frac{(J^+)^k}{k!}[y^a{}_2F_1(a,b;c;x)]$$

$$= \sum_{k=0}^{\infty}\frac{(J^+)^{k-1}}{k!}[ay^{a+1}{}_2F_1(a+1,b;c;x)]$$

$$= \sum_{k=0}^{\infty}\frac{(J^+)^{k-k}}{k!}[(a)_k y^{a+k}{}_2F_1(a+k,b;c;x)]$$

Thus,

$$e^{J^+}[y^a{}_2F_1(a,b;c;x)] = \sum_{k=0}^{\infty}\frac{(a)_k}{k!}[y^{a+k}{}_2F_1(a+k,b;c;x)] \tag{4.9}$$

Equating the two equations (4.8) and (4.9), we get

$$(\frac{1}{1-y})^a{}_2F_1(a,b;c;\frac{x}{1-y}) = \sum_{k=0}^{\infty}\frac{(a)_k}{k!}{}_2F_1(a+k,b;c;x)y^k. \tag{4.10}$$

Finally, we want to point out that the interchanging of the order of the operators in (3.6) will give different results.

5. Particular Cases

We can obtain the following generating function relations for the hypergeometric polynomials.

I: Setting $b=-n$ in (4.7), we get

$$(1-t)^{a-n-c}(1-t(1-x))^n{}_2F_1(a,-n;c;\frac{x}{(1-t(1-x))}) \tag{5.1}$$

$$=\sum_{m=0}^{n}\frac{(c-a)_m}{m!}[{}_2F_1(a-m,-n;c;x)t^m].$$

I: Setting $b=-n$ in (4.10), we get

$$(\frac{1}{1-y})^a{}_2F_1(a,-n;c;\frac{x}{1-y}) = \sum_{k=0}^{n}\frac{(a)_k}{k!}{}_2F_1(a+k,-n;c;x)y^k. \tag{5.2}$$

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