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Ternary Permutable Semigroups of the First Kind

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Abstract: *A semigroup S is called a permutable semigroup if $\rho\cdot\sigma=\sigma\cdot\rho$ is satisfied for all congruences ρ and σ of S .A non empty set S together with a ternary multiplication denoted by juxtaposition, is said to be a ternary semigroup if $(abc)de=a(bcd)e=ab(cde)$ for all a,b, c, d, e $\in S$. In this paper we deal with permutable ternary semigroups of the first kind.*

Keywords: *Ternary semigroup, permutable semigroup.*

1. Introduction

The first paper about permutable semigroups is [4] where some general theorems are proved and the commutative permutable semigroups are described.

Using the terminology of [5], a semigroup S is called a semigroup of type A if it is a semilattice of a nil semigroup S_0 and a rectangular group $S_1=L\times G\times R$ with $|L|\le 2,|R|\le 2(L$ is a left zero semigroup , G is a group , R is a right zero semigroup).A semigroup S of type A is called of the first kind if $a\in S_1aS_1$ for every $a\in S$.

Let S be a ternary permutable semigroup of the first kind. Then S is a semilattice of a nil semigroup S_0 and a rectangular abelian group $S_1=L\times G\times R$ with $|L|\le 2,|R|\le 2(L$ is a left zero semigroup , G is a group , R is a right zero semigroup). It is obvious that S_1 is a rectangular band $L\times R$ of discoint subgroups $G_{ij}=\{i\}\times G\times\{j\}(i\in L, j\in R)$ and the idempotent elements of S_1 are the identity elements $e_{ij}=(i,e,j)$ of G_{ij} (here e denotes the identity element of G).

Introduce the following notation: for an element t of a non-empty set T containing at most two elements, let $\bar{t}=t$ if $|T|=1$ and let $\bar{t}\in T-\{t\}$ if $|T|=2$.

Definition1:A semigroup S is called a permutable semigroup if $\rho\cdot\sigma=\sigma\cdot\rho$ is satisfied for all congruences ρ and σ of S .

Definition2: A non empty set S together with a ternary multiplication denoted by juxtaposition, is said to be a ternary semigroup if $(abc)de=a(bcd)e=ab(cde)$ for all $a,b,c,d,e\in S$.

Definition3: A ternary semigroup S is said to be commutative if $x_1x_2x_3=x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ for every permutation σ of $\{1,2,3\}$ and $x_1,x_2,x_3\in S$.

2. Ternary Permutable Semigroups of the First Kind

Lema 1: If S is a ternary permutable semigroup of the first kind then, for every $a \in S, i \in L$ and $j \in R$ we have

(i) $e_{ij}a = e_{i\bar{j}}a$.

(ii) $ae_{ij} = ae_{\bar{i}j}$.

Proof. As S is ternary permutable semigroups for every $a \in S, i \in L$ and $j \in R$ we have

$$e_{ij}a = e_{ij}e_{i\bar{j}}e_{ij}a = e_{ij}e_{ij}e_{i\bar{j}}a = e_{i\bar{j}}a$$

and

$$ae_{ij} = ae_{ij}e_{\bar{i}j}e_{ij} = ae_{\bar{i}j}e_{ij}e_{ij} = ae_{\bar{i}j}. \ \square$$

Introduce the following notations. For arbitrary $i \in L$ and $j \in R$, let

$A_i = e_{ij}S = e_{i\bar{j}}S$ and $B_j = Se_{ij} = Se_{\bar{i}j}$.

It is clear that $A_i = G_{ij} \cup G_{i\bar{j}} \cup e_{ij}S_0$ and $B_j = G_{ij} \cup G_{\bar{i}j} \cup S_0 e_{ij}$.

A semigroup is said to be left (right) commutative if it satisfies the identity $abc = bac(abc = acb)$.

Lema 2: Let S be a ternary permutable semigroup of the first kind. Then $A_i (i \in L)$ and $B_j (j \in R)$ are left and right commutative subsemigroups of S, respectively.

Proof. It is clear that e_{ij} is left identity elements of A_i. Then, for arbitrary elements $a, x, y \in A_i$,

$$xya = e_{ij}xya = e_{ij}yxa = yxa.$$

Hence A_i is left commutative. The proof of the assertion for B_j is similar. $\square$

Lemma 3: Let S be a ternary permutable semigroup of the first kind. Then

$$S = A_i \cup A_{\bar{i}} = B_j \cup B_{\bar{j}} (i \in L, j \in R).$$

Moreover, $A_i \cap A_{\bar{i}}$ and $B_j \cap B_{\bar{j}}$ $(i \in L, j \in R)$ are ideals of S.

Proof. Let S be a ternary permutable semigroup of the first kind. Then for every $a \in S$ there is an element $e_{ij} \in E(S_1)$ such that $a = e_{ij}a \in A_i$.

Thus $S = A_i \cup A_{\bar{i}}(i \in L)$. Similarly, $S = B_j \cup B_{\bar{j}}(j \in R)$.

It is clear that $A_i \cap A_{\bar{i}} \neq \varnothing$ is a right ideal of S. Let $s \in S, a \in A_i \cap A_{\bar{i}}$ be arbitrary elements. Then $e_{t,k}a = a$ for every $t \in L, k \in R$. Assume $s \in A_i$. As A_i is a subsemigroup of $S, sa \in A_i$. As S is of the first kind, $a = at$ for an element $t \in S_t$.

Thus for arbitrary $j \in R$, $e_{\bar{i}j}sa = e_{\bar{i}j}sat = e_{i\bar{j}}ast = ast = e_{ij}ast = e_{ij}sat = sa$, that is $sa \in A_{\bar{i}}$.

Thus $sa \in A_i \cap A_{\bar{i}}$. Hence $A_i \cap A_{\bar{i}}$ is an ideal of A_i.We can similarly prove that $A_i \cap A_{\bar{i}}$ is an ideal of $A_{\bar{i}}$. Hence $A_i \cap A_{\bar{i}}$ is an ideal of S. The proof of the assertion that $B_j \cap B_{\bar{j}}$ is an ideal of S is similar. □

Lema 4: If f is an idempotent element of a ternary semigroup S, then

$\eta_f = \{(x,y) \in S \times S \mid fx = fy\}$ and $\mu_f = \{(x,y) \in S \times S \mid xf = yf\}$ are congruences on S.

Proof. It is clear that η_f is a right congruence. Let x, y, z be arbitrary elements of S such that $(x,y) \in \eta_f$. Then $fzx = ffzx = fzfx = fzfy = ffzy = fzy$ and so $(sx, sy) \in \eta_f$.

Hence η_f is a congruence on S. The proof is similar for μ_f.□

Lema 5: If S is a ternary permutable semigroup of the first kind then, for every $i \in L$ and $j \in R$

(1) $\eta_{e_{i\bar{j}}} = \eta_{e_{ij}} = \eta_{e_{\bar{i}\bar{j}}} = \eta_{e_{\bar{i}j}}$.

(2) $\mu_{e_{ij}} = \mu_{e_{\bar{i}\bar{j}}} = \mu_{e_{i\bar{j}}} = \mu_{e_{\bar{i}j}}$.

Proof. By lema 1, $\eta_{e_{i\bar{j}}} = \eta_{e_{ij}}$ and $\eta_{e_{\bar{i}\bar{j}}} = \eta_{e_{\bar{i}j}}$.

We show that $\eta_{e_{\bar{i}j}} = \eta_{e_{ij}}$.

Assume that $(a,b) \in \eta_{e_{ij}}$ for some $a, b \in S$. Then $e_{ij}a = e_{ij}b$ and so

$$e_{\bar{i}\bar{j}}a = e_{\bar{i}\bar{j}}e_{ij}a = e_{\bar{i}\bar{j}}e_{ij}b = e_{\bar{i}\bar{j}}b.$$

Then $(a,b) \in \eta_{e_{\bar{i}j}}$. Thus, $\eta_{e_{ij}} \subseteq \eta_{e_{\bar{i}j}}$. Similarly $(a,b) \in \eta_{e_{\bar{i}j}}$ for some $a, b \in S$, then $e_{\bar{i}j}a = e_{\bar{i}j}b$

and so $e_{ij}a = e_{ij}e_{\bar{j}\bar{i}}a = e_{ij}e_{\bar{j}\bar{i}}b = e_{ij}b$. Then $(a,b) \in \eta_{e_{ij}}$. Thus $\eta_{e_{\bar{i}j}} \subseteq \eta_{e_{ij}}$. Hence $\eta_{e_{\bar{i}j}} = \eta_{e_{ij}}$ ((1) is satisfied). The proof of (2) is similar. □

Lema 6: If S is a ternary permutable semigroup of the first kind then for every $i \in L$ and $j \in R, A_i \cong S/\eta$ and $B_j \cong S/\mu$.

Proof. Let $[a]_\eta$ denote the η-class of S containing the element a of S. We show that

$[a]_\eta = \left(E\left(S_1\right)\right)a$. Assume $\left(x, y\right) \in \eta$ for some $x, y \in A_i$. As e_{ij} is a left identity element of A_i, we have $x = e_{ij}x = e_{ij}y = y$. Thus $\eta / A_i = id_{A_i}$ where η / A_i is the restriction of η to A_i and id_{A_i}

Is the identity relation of A_i. Let $a \in S$ be an arbitrary element. Then by lema3, $S = A_i \cup A_{\bar{i}}$, and so there is an element $i \in L$ such that $a \in A_i$. As $e_{ij}a = e_{ij}e_{ija}$ $j \in R$, we have $(a, e_{ij}a) \in \eta$.

Thus $[a]_\eta = \left\{a, e_{\bar{i}j}a\right\}$.

Since $a = e_{ij}a = e_{\bar{i}\bar{j}}a$ and $e_{\bar{i}\bar{j}}a = e_{\bar{i}\bar{j}}e_{ij}e_{\bar{i}\bar{j}}a = e_{\bar{i}\bar{j}}e_{\bar{i}\bar{j}}e_{ij}a = e_{\bar{i}j}a$, we get $[a]_\eta = \left\{a, e_{\bar{i}j}a\right\} = (E(S_1))a$.

This result implies that $|A_i \cap [a]_\eta| = 1$ for every $a \in S$. Let Φ_i denote the mapping of S/η to A_i defined by $\Phi_i : [a]_\eta \to A_i \cap [a]_\eta$. Then Φ_i is bijective. As $(A_i \cap [a]_\eta)(A_i \cap [b]_\eta) \in (A_i \cap [ab]_\eta)$ we get $\Phi_i(a)\Phi_i(b) = (A_i \cap [a]_\eta)(A_i \cap [b]_\eta) \in (A_i \cap [ab]_\eta) = \Phi_i(ab)$ which means that Φ_i is a homomorphism.

Thus Φ_i is an isomorphism of S/η onto A_i. The proof of $B_j \cong S/\mu$ is similar □.

Corollary 1: Let S be a ternary permutable semigroup of the first kind. Then for every $i \in L$ and $j \in R$, $\phi_i : a \to a' = e_{\bar{i}j} a (a \in A_i)$ and $\Psi_j : b \to b' = b e_{i\bar{j}} (b \in B_j)$ are isomorphisms of A_i and B_j onto $A_{\bar{i}}$ and $B_{\bar{j}}$, respectively.

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