International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 7, Issue 8, 2019, PP 1-2
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0708001
www.arcjournals.org

A matrix trace inequality

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Abstract: *In this short paper, we give a affirmative answer on matrix trace inequalities for the product of positive semidefinite matrices under a condition.*

Keywords: *Matrix trace inequality Majorization Positive semidefinite matrix Contractive.*

1. Introduction

We give some notations, Let $M_{m\times n}(C)$ be the space of all complex matrices of size $m\times n$, For $A\in M_n(C)$, the vector of eigenvalues of A is denoted by $\lambda(A)=(\lambda_1(A),\lambda_2(A),...,\lambda_n(A))$ If A is Hermitian, we arrange the eigenvalues of A in nonincreasing order, $\lambda_1(A)\ge\lambda_2(A)\ge...\ge\lambda_n(A)$.In addition $x\prec y$ means that $x=(x_1,x_2,...,x_n)$ is majorized by $y=(y_1,y_2,...,y_n)$ with $x_1\ge x_2\ge...\ge x_n$ and $y_1\ge y_2\ge...\ge y_n$, if we have $\sum_{i=1}^{k}x_i\le\sum_{i=1}^{k}y_i$ $(k=1,...,n-1)$ and $\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}y_i$.

$x\prec_w y$ means that $x=(x_1,x_2,...,x_n)$ is weak majorized by $y=(y_1,y_2,...,y_n)$, if we have $\sum_{i=1}^{k}x_i\le\sum_{i=1}^{k}y_i$ $(k=1,...,n)$.

The purpose of this paper is to give the answer to the following problem which was given in the paper [1].

Problem1.1 For A, $B\in M_n(C)$ are positive semidefinite matrices and K is contraction, the following inequality hold or not?

In fact,we easily find that the inequality isn't true if A, B is noncommutative, For example, Let

$A=\begin{bmatrix}10&0\\0&1\end{bmatrix}$, $B=\begin{bmatrix}1&3\\3&10\end{bmatrix}$, $K=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $AB\ne BA$, then $|trAKB|=33>tr(AC)=20$.

2 Main resluts

Lemma1 Denoted the eigenvalues of matrix A by $\lambda(A)=(\lambda_1,...,\lambda_n)$, then $\{|\lambda_i|\}_{i=1}^{n}\prec_w s(A)$.

Specially , $|trA|\le\sum_{i=1}^{n}s_i(A)$.

Lrmma2 $A_1,\ ...,A_m$ are n-square complex matrices, then $s(\prod_{j=1}^{m}A_j)\prec_w\left\{\prod_{j=1}^{m}s_i(A_j)\right\}_{i=1}^{n}$.

Theorem3 $A, B \in M_n(C)$ are positive semidefinite matrices and K is contraction and $AB = BA$, then $|trAKB| \leq trAB$.

Proof $A, B \in M_n(C)$ are positive semidefinite matrices and $AB = BA$, so AB is positive semidefinite.

According to Lemma2, $s(BAK) \prec s(BA) = \lambda(BA)$. Then

$$|trAKB| = |trBAK| \leq \sum_{i=1}^{n} s_i(BAK) \leq \sum_{i=1}^{n} s_i(BA) = \sum_{i=1}^{n} \lambda_i(BA) = trBA = trAB$$

So the theorem is completed.

Corollary4 $H = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix}$ is positive semidefinite, $AC = CA$, then for any integer m, $|trB^m| \leq tr(A^{\frac{m}{2}} C^{\frac{m}{2}})$.

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Citation: *Feng Zhang & Jinli Xu (2019). A matrix trace inequality. International Journal of Scientific and Innovative Mathematical Research (IJSIMR), 7(8), pp. 1-2. http://dx.doi.org/ 10.20431/2347 -3142.0708001*