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Note on the Vector Sub-Space $K(H)^2$ of Compact Operators on a H Hilbert's Infinite Dimension Separable Space

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Abstract: *can we provide a vector space with two Hilbert's space structures? Obviously, the vector space $K(H)$ of compact operators is candidate. The targeted vector spaces are noted $K(H)^1$ and $K(H)^2$. The following lines present the $(H)^2$ space.*

Keywords: *vector space, scalar products, norm, full norm, hilbertian basis, Cauchy-Schwarz inequality, compact operators, Banach's space, Hilbert's space.*

1. Preliminieries

1.1. Information

The vector space $\mathrm{B}(H)$ of operators limited on Hilbert's H infinite dimension separable space is Hilbert's space of which the scalar product and the full hermitian norm are respectively noted in this way:

$< A, B >_1 = \sum_{i=1}^{\infty} \frac{1}{2^i} \left(A_{e_i}, B_{e_i}\right)$ for ale A, B ∈ $B(H)$, $e_i \in b$ and

$\|A\|_1 = \left(\sum_{i=1}^{\infty} \frac{1}{2^i} \|Ae_i\|^2\right)^{1/2}$ for each A∈ $\mathrm{B}(H)$, $e_i \in b$, b being a hilbertian basis. Sometimes, we will note these results respectively by $<,>_1$ *and* $\| \quad \|_1$.

[*Masamba Sala Voka Joseph*]

2. Candidate $K(H)$ Vector Space

As the question asked alludes to the existence of two compact vector spaces, we have found it appropriate to call and note them respectively $K(H)^1$ and $K(H)^2$. These notations do not add nor lessen the properties of a compact space $K(H)$; they inherit from all the properties due to a compact space noted $K(H)$. The following lines speak about the vector space $K(H)^2$.

3. Recalls

3.1. Definition

Consider A an operation attached to a separable Hilbert's space H. It is said that A is a compact operator if it changes any limited part D of part H into a pre-compact part $A(D)$, whereas A is a finite rank dimension operator if its image $R(A)$ is finite.

3.2. Proposition

Consider H a separable Hilbert's space and (A_n) a succession of compact operators which converge towards a linear operator A, then a compact operator.

3.3. Proposition

Consider H a separable Hilbert's space and A a compact operator; then there exists a succession (A_n) of finite rank operators such as $\lim_{n\to\infty} A_n = \mathrm{A}$ [*Dieudonné*]

4. NOTE

As $K(H)^2$ is a closed vector sub–space of B(H), it benefits from properties of Hilbert's space B(H) such as :

- The closed unity bowl is noted $B_H = x \in H : \|x\| \leq 1$
- The following proposition : for a compact operator A and a vector x $\in B_H$ we have the inequality: $\|Ax\|^2 \leq (\sum_{i=1}^{\infty}|a_i|\|Ae_i\|)^2$
- The field of definition of a compact operator A is dense in H and for all the two compact operators A, B $\in K(H)^2$ and $e \in$ b, the scalar $(\mathrm{Ae_i}\ , \mathrm{Be_i})$ is an element of ℂ.

5. REMARK

$K(H)^2$ is a vector sub-space of B(H) so that the field of the definition of compact operators including those that are the terms of a succession (A_n) of finite rank operators, are denses in H.

Thus, whatever an operator A and a term A_n of a succession (A_n) of finite rank operators such as $\lim_{n\to\infty} A_n = A$ for any n ∈ ℕ, $\overrightarrow{D(A)} = H$ and $\overrightarrow{D(A_n)} = H$.

Now consider A and B two compact operators such as $\lim_{n\to\infty} A_n = A,\ \lim_{n\to\infty} B_n = B$ and a vector $e_i \in b$. It is clear that $A_n\, e_i\ and\ B_n\, e_i$ are vectors of H since $A_n\ and\ B_n$ are operators on H. it follows that $(A_n e_i\ , B_n e_i)$ is a scalar.

For this particular case of compact operators, we will use in the serial, the terms of two successions of finite rank operators (A_n) and (B_n) such as $\lim_{n\to} A_n = A$ and $\lim_{n\to} B_n = B$ whatever two compact operators A and B.

Concretely, if for A and B we take respectively $A_n\ and\ B_n$, then the number $< A, B >_1 = \sum_{i=1}^{\infty}\frac{1}{2^i}(Ae_i, Be_i)$ takes the following form : $< A, B >_2 = \sum_{i=1}^{\infty}\frac{1}{2^i}(A_n e_i, B_n e_i)$ for all the two operators A, B on H and $e_i \in b$, b being a Hilbertian basis. It is wise to be sure that this formula does not depend on the choice of finite rank operators $(A_n)\ and\ (B_n)$ such as : $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$.

In fact, if $(P_n)\ and\ (Q_n)$ are two other successions of finite rank operators such as $\lim_{n\to\infty} P_n = A$ and $\lim_{n\to\infty} Q_n = Q$, it is clear that if $\lim_{n\to\infty} P_n = A$ and $\lim_{n\to\infty} Q_n = B$, then it must absolutely get the following equalities :

$A = P\ and\ B = Q$. These equalities are true since the limit of a succession is unique; they involve the equalitie $< A, B > = < P, Q >$ that clearly means that: $\sum_{i=1}^{\infty}\frac{1}{2^i}(A_n e_i, B_n e_i) = \sum_{i=1}^{\infty}\frac{1}{2^i}(P_n e_i, Q_n e_i)$.

We have found it appropriate to name this scalar product in this way : $< A, B >_2 = \sum_{n=1}^{\infty}\frac{1}{2^n}$ $(A_n e_i, B_n e_i)$ for all two compact operators A , B and $e_i \in b$; sometimes, we note it simply : $<, >_2$.

6. STRUCTURES CONFERRED TO A COMPACT VECTOR SPACE $K(H)^2$

6.1. Theorem

Consider $K(H)^2$ Hilbert's infinite dimension separable complex spaces, A and B two compact operators such as A = $\lim_{n\to} A_n$, $B = \lim_{n\to\infty} B_n$, $e_i \in b$, and the following application : $<,>_2 :(K(H)^2)^2 \to ℂ$ such as $< A, B >_2 = \sum_{i=1}^{\infty}\frac{1}{2^i}(Ae_i, Be_i\ )$ for A , B $\in K(H)^2$ and $e_i\ \in b$; then $<,>_2$ is a scalar product on K$(H)^2$.

PROOF

(i) Whatever three compact operators A, B, C $\in K(H)$ and $e_i \in b$:

$(i_1) < A + B, C >_2 =$

$$= \sum_{n=1}^{\infty}\frac{1}{2^n}(\ A_n e_i, C_n e_i) + \sum_{n=1}^{\infty}\frac{1}{2^n}(\ B e_i, C_n e_i)$$

$$=< A,C >_2 + < B,C >_2$$

$(i_2) < A,B+C >_2 =$

$$= \sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, B_n e_i) + \sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, C_n e_i)$$

$$=< A,B >_2 + < A,C >_2$$ \

(ii) Whatever two compact operators A, B , a scalar t and $e_i \in b$:

(ii_1) $< tA,B >_2 =$

$$= \sum_{n=1}^{\infty}\frac{1}{2^n}((tA_n)e_i, B_n e_i) = \sum_{n=1}^{\infty}\frac{1}{2^n}t(A_n e_i, B_n e_i)$$

$$= t\sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, B_n e_i) = t\ < A,B >_2$$

$(ii_2) < A,tB >_2 \ =$

$$= \sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, (tB_n)e_i) = \sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, tB_n e_i)$$

$$= \vec{t}\sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, B_n e_i) = \vec{t} < A,B >_2$$

(iii) Whatever two compact operators A, B and $e_i \in b$:

$< A,B >_2 =$

$$= \sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, B_n e_i) = \sum_{n=1}^{\infty}\frac{1}{2^n}\overrightarrow{(B_n e_\iota,\ A_n e_\iota\,)}$$

(iv) Whatever a compact operator A and $e_i \in b$:

$< A,A >_2 \ = \sum_{n=1}^{\infty}\frac{1}{2^n}\ \|A_n e_i\|^2 > 0$; it follows that $\|A_n e_i\|^2 \ > 0$ for any $e_i \in b$. the equality $< A,A >_2 \ = 0$ means that we have $< A,A >_2 = \sum_{n=1}^{\infty}\frac{1}{2^n}(A_n e_i, B_n e_i) = \sum_{n=1}^{\infty}\frac{1}{2^n}\ \|A_n e_i\|^2 = 0$ whatever $e_i \in b$ or simply $A = 0_H$.

The norm associated with the scalar productions is therefore noted : $\|A\|_2 = \left(\sum_{n=1}^{\infty}\frac{1}{2^n}\|A_n e_i\|^2\right)^{1/2}$ for any compact operator A and $e_i \in b$. ■

7. Remark

Consider the two norms $\|\ \ \|_2$ and $\|\ \ \|_1$ on $K(H)^2$; in this work, the word 'norm" means a structure of general topology or functional analysis on the one hand, on the other hand in arithmetic's, a real number on the other hand: for instance: $\|\ \ \|_1 \le \|\ \ \|_2$ and $\|\ \ \|_2 \le \|\ \ \|_1$ which means that : $\|\ \ \|_1 = \|\ \ \|_2$.
These elements, the proposition (2) of (4 Note) and the comparison of norms have helped enough us to get the desired results:

8. Comparison of Norms $\|\ \ \|_1$ *and* $\|\ \ \|_2$

8.1. Theorem

Consider H a separable Hilbert's space on k and $\|\ \ \|_1$ *and* $\|\ \ \|_2$ the two norms on the compact vector space $K(H)^2$, then we have : $\|\ \ \|_1 \le \|\ \ \|_2$.

Proof

In fact, for any compact operator A and a vector x of H belonging to B_H, we have the following equalities and inequalities:

$\|Ax\|^2 \ \le (\sum_{i=1}^{\infty}|a_i|\ \|Ae_i\|)^2$

$[Porposition\ (2)\ \text{the 4. note}]$

$\le\ (\sum_{i=1}^{\infty}|a_i|\|Ae_i\|_2)^2 = (\sum_{i=1}^{\infty}|a_i|\ \ \|A\|_2\ \|e_i\|)^2$

$[since\ \|A\|_2 = sup\{\|Ae_i\|: i = 1,2,3,\dots\}]$

$= \sum_{i=1}^{\infty}|a_i|^2\|A\|_2^2 = \|A\|_2^2$

$[since\ \sum_{i=1}^{\infty}|a_i|^2 = 1 = \|e_i\|^2\ ]$

Brief :

$\|Ax\|^2 \leq\ \|A\|_2^2$ or simply $\|Ax\| \leq \|A\|_2$; it follows that the norm

$\|A\|_1\ \ = sup\{\|Ax\|: \|x\|\ \leq 1\}\ \ \leq \|A\|_2$ or simply the result:

$\|A\|_1\ \ \leq \|A\|_2.$■

8.2. Theorem

Consider H a separable Hilbert's space on $\mathbb{K}$ and $\|\ \ \|_1\ and \|\ \ \|_2$ the two numbers on the vector space $K(H)^2$: then, we have : $\|\ \ \|_2\ \leq \|\ \ \|_1$.

PROOF

Il is easy to note that if A is a compact operator and x a vector of H belonging to B_H, we have the following inequalities and equalities:

$\|A\|_2^2 = \sum_{i=1}^{\infty}\frac{1}{2^i}(Ae_i, Ae_i)$ [by definition of $\|\ \ \|_2$]

$\leq \sum_{i)1}\frac{1}{2^i}|(Ae_i, Ae_i)|\ \leq \sum_{i=1}^{\infty}\frac{1}{2^i}\ \|Ae_i\|Ae_i\|\|$

[Cauchy − Schwarz's inequality]

$= \sum_{i)1}^{\infty}\frac{1}{2^i}\ (\|Ae_i\|)^2 \leq\ \sum_{i=1}^{\infty}\frac{1}{2^i}\|Ae_i\|_1^2$

[since $\|A\|_1 = sup\{\|Ae_i\|: e_i \in b\}$]

$= \|A\|_1^2\ \|e_i\|^2\ \sum_{i=1}^{\infty}\frac{1}{2^i}$

$=\ \|A\|_1^2$ $\left[\text{since}\ \ \sum_{i=1}^{\infty}\frac{1}{2^i} = 1 = \|e_i\|^2\right]$

finally $\|A\|_2^2 \leq \|A\|_1^2$ or simply $\|A\|_2 \leq \|A\|_1$.■

9. CONCLUSION

9.1. Remark

According as we consider a norm a numerical number, a topological structure, a functional analysis structure and, given results got, we present the conclusion in these words:

9.2. Theorem

The inequalities $\|\ \ \|_1\ \leq \|\ \ \|_2$ and $\|\ \ \|_2\ \leq \|\ \ \|_1$ mean that the two numbers $\|\ \ \|_1\ and\ \|\ \ \|_2$ are equal or simply : $\|\ \ \|_1\ = \|\ \ \|_2$. ■

9.3. Theorem

The two norms $\|\ \ \|_1$ and $\|\ \ \|_2$ are hermitian and full; the vector space $K(H)^2$ provided with the norm $\|\ \ \|_2$ is a Banach's space and, also the vector space $K(H)^2$ provided with the norm $\|\ \ \|_2$ is a Banach's space. ■

9.4. Theorem

The two norms $\|\ \ \|_1$ and $\|\ \ \|_2$ are hermitian and full; the vector space $K(H)^2$ provided with the norm $\|\ \ \|_2$ is a Hilber's space and, also the vector space $K(H)^2$ provided with the norm $\|\ \ \|_2$ is a Hilbert's space. ■

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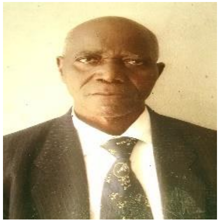

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