International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 7, Issue 5, 2019, PP 10-13
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0705003
www.arcjournals.org

Some Results of Real Symmetric Semi-Definite Matrix Traces

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Abstract: *The first part of this paper is to explore the inequalities of the traces of real symmetric semi-positive definite matrices under partial order relations. We find more general conclusions from special results*

Keywords: *Semi-definite matrix Trace inequality Partial ordering*

1. Main Results

In the partial order relationship, we calculate the product of the semi-positive definite matrices with smaller powers, and compare the size relationship.

If A and B are real symmetric semi-positive definite matrices (this part is discussed under the condition of real symmetric semi-positive definite matrices, and we will not

Emphasize it later), $A \geq B$, then $\mathrm{tr}\,(A^2 - AB) = \mathrm{tr}\,A\,(A - B) \geq 0$, $\mathrm{tr}\,(AB - B^2) = \mathrm{tr}\,(A - B)\,B \geq 0$.

So, we get

$\mathrm{tr}A^2 \geq \mathrm{tr}AB \geq \mathrm{tr}B^2$. Next, we see that

$\mathrm{tr}\,(A^2 B - AB^2) = \mathrm{tr}\,[AB\,(A - B)] = \mathrm{tr}\,B\,(A - B)\,A = \mathrm{tr}\,B\,(A - B)\,B + \mathrm{tr}\,B\,(A - B)\,(A - B) \geq 0$,

Then we prove $\mathrm{tr}A^2 B \geq \mathrm{tr}AB^2$.

And it is not difficult to calculate that

$\mathrm{tr}A^3 - \mathrm{tr}A^2 B = \mathrm{tr}A^2\,(A - B) \geq 0$, $\mathrm{tr}AB^2 - \mathrm{tr}B^3 = \mathrm{tr}\,(A - B)\,B^2 \geq 0$.

So we can get a sequence relationship that $\mathrm{tr}A^3 \geq \mathrm{tr}A^2 B \geq \mathrm{tr}AB^2 \geq \mathrm{tr}B^3$.

$$tr(A^3B - AB^3) = tr(A^2B - AB^2)(A+B) = tr(A-B)B(A+B)A$$

$$= tr(A-B)B(A+B)B + tr(A-B)B(A+B)(A-B)$$

$$= tr(A-B)B(A+B)B + trB(A+B)(A-B)^2 = tr(A-B)B(A+B)B + trB^2(A-B)^2 + trBA(A-B)^2$$

$$= tr(A-B)B(A+B)B + trB^2(A-B)^2 + trA(A-B)^2B$$

$$= tr(A-B)B(A+B)B + trB^2(A-B)^2 + trB(A-B)^2B + tr(A-B)^3B \geq 0$$

So $trA^3B \geq trAB^3$

$$tr(A^2B^2 - AB^3) = trBA(A-B)B = trB^3(A-B) + trB^2(A-B)^2.$$

According to $B^3\,(A - B)$, $B^2\,(A - B)^2$ are similar to the non-negative diagonal matrix, so $trB^3(A-B)+trB^2(A-B)^2 \geq 0$，于是可以得到 $tr(A^2B^2-AB^3)\geq 0$

$$tr(A^3B-A^2B^2)=trA(A-B)BA$$

$$=trB(A-B)BB+trB(A-B)B(A-B)+tr(A-B)^2BA$$

We can get $trB(A-B)BB\geq 0$， $trB(A-B)B(A-B)\geq 0$ by $A\geq B$.

And $tr(A-B)^2BA=trA(A-B)^2B=trB(A-B)^2B+tr(A-B)^3B\geq 0$， so we say $tr(A^3B-A^2B^2)\geq 0$.

So we can get that $trA^3B\geq trA^2B^2\geq trAB^3$， we use the same methods can prove $trA^4\geq trA^3B\geq trA^2B^2\geq trAB^3\geq trB^4$.

$$tr(A^4B-AB^4)=tr(A^2B-AB^2)(A^2+B^2)+tr(A^3B^2-A^2B^3)$$

$$tr(A^2B-AB^2)(A^2+B^2)=tr(A-B)B(A^2+B^2)A$$

$$=tr(A-B)B(A^2+B^2)B+trBA^2(A-B)^2+trB^3(A-B)^2$$

$tr(A^3B^2-A^2B^3)=trB^2(A-B)A^2$,

So $tr(A^4B-A^3B^2)\geq 0$,

For $tr(A^4B-A^3B^2)=trA^3(A-B)B=trAB(A-B)BA+trA(A-B)^2BA$

And $trA(A-B)^2BA=trB(A-B)^2BA+tr(A-B)^3BA$

$$=trB(A-B)^2BA+trB(A-B)^3B+tr(A-B)^4B\geq 0$$

Then $tr(A^4B-A^3B^2)=trA^3(A-B)B=trAB(A-B)BA+trA(A-B)^2BA\geq 0$

Finally ,we know that $trBA^2(A-B)^2+trB^2(A-B)A^2=tr(A^4B-A^3B^2)\geq 0$

Taking these circumstances, we get that $tr(A^4B-AB^4)\geq 0$.

Some of the above formulas give us inspiration:

If s + t = k， A, B are real symmetric positive definite matrices (guaranteed inequality meaningful) A ≥ B, Is there $trA^k \geq trA^{k-1}B \geq trA^{k-2}B^2 \geq .... \geq trAB^{k-1} \geq trB^k$ established？

Let us give a positive answer to this question.

Theorem If s + t = k , s ,t ∈ N , A, B are real symmetric positive definite matrices , A ≥ B , Then $trA^k \geq trA^{k-1}B \geq trA^{k-2}B^2 \geq .... \geq trAB^{k-1} \geq trB^k$ is established.

Let m be an integer greater than 1, A, B are real symmetric positive definite matrices, then $tr(A^m B) \geq tr(A^{m-1}B^2)$

$tr(A^m B) - tr(A^{m-1}B^2) = tr[A^{m-1}(A-B)B] = tr[A^{m-2}B(A-B)B] + tr[A^{m-2}(A-B)^2 B]$

$tr[A^{m-2}B(A-B)B] \geq 0$, If $tr[A^{m-2}(A-B)^2 B] \geq 0$, Then the proposition can be proved.

$tr[A^{m-2}(A-B)^2 B] = tr[A^{m-3}B(A-B)^2 B] + tr[A^{m-3}(A-B)^3 B]$, From the equation, we realize that we should prove that $tr[A^{m-3}(A-B)^3 B] \geq 0$, Decompose for $tr[A^{m-3}(A-B)^3 B]$, And summarize it, finally we only prove $tr[A(A-B)^{m-1} B] \geq 0$, and $tr[A(A-B)^{m-1} B] = tr[B(A-B)^{m-1} B] + tr[(A-B)^m B] \geq 0$.

So we get that $tr(A^m B) \geq tr(A^{m-1}B^2)$.

$tr(A^{m-1}B^2) - tr(A^{m-2}B^3) = tr[A^{m-2}(A-B)B^2]$,

$tr[A^{m-2}(A-B)B^2] = tr(A-B)A^{m-3}(A-B)B^2 + trBA^{m-3}(A-B)B^2$

$tr(A-B)A^{m-3}(A-B)B^2 \geq 0$ can be obtained from the properties of semi-definite matrix traces.

Next, $trBA^{m-3}(A-B)B^2 = trA^{m-3}(A-B)B^3 = tr(A-B)A^{m-4}(A-B)B^2 + trBA^{m-4}(A-B)B^2$

We repeat the process above, and we can see $tr(A^{m-1}B^2) \geq tr(A^{m-2}B^3)$ by $trB(A-B)B^2 \geq 0$.

Use the same way, we also can prove $tr(A^{m-2}B^3) \geq tr(A^{m-3}B^4)$ ...

We assume that $s \geq 1$ is a positive integer, then $tr(A^{s+1}B^s) \geq tr(A^s B^{s+1})$

For $tr(A^{s+1}B^s) - tr(A^s B^{s+1}) = trA^s(A-B)B^s$

$$= tr(A-B)A^{s-1}(A-B)B^s + trBA^{s-1}(A-B)B^s$$

$= tr(A-B)A^{s-1}(A-B)B^{s} + tr(A-B)A^{s-2}(A-B)B^{s+1} + trBA^{s-2}(A-B)B^{s+1}$

$trBA^{s-2}(A-B)B^{s+1} = tr(A-B)A^{s-3}(A-B)B^{s+2} + trBA^{s-3}(A-B)B^{s+2}$

We repeat this process over and over again,

then we can see $trBA(A-B)B^{2s-1} = tr(A-B)(A-B)B^{2s} + trB(A-B)B^{2s}$, and

$tr(A-B)(A-B)B^{2s} = tr(A-B)B^{2s}(A-B) \geq 0$, $trB(A-B)B^{2s} = tr(A-B)B^{2s+1} \geq 0$,

Then we prove that $tr(A^{s+1}B^{s}) \geq tr(A^{s}B^{s+1})$.

So we can prove that $trA^{k} \geq trA^{k-1}B \geq trA^{k-2}B^{2} \geq .... \geq trAB^{k-1} \geq trB^{k}$.

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Citation: *Feng Zhang & Jinli Xu (2019). Some Results of Real Symmetric Semi-Definite Matrix Traces. International Journal of Scientific and Innovative Mathematical Research (IJSIMR), 7(5), pp.10-13. http://dx.doi.org/ 10.20431/2347-3142.0705003*