International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 7, Issue 4, 2019, PP 28-31
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0704005
www.arcjournals.org

Some Inequalities on Semi-Positive Definite Matrices

Feng Zhang*, Jinli Xu

Department of mathematics, Northeast Forestry University, Harbin 150040, China

Corresponding Author:** ***Feng Zhang, *Department of mathematics, Northeast Forestry University, Harbin 150040, China*

Abstract: *The paper mainly introduces some inequalities of semi-positive definite matrices under partial order relations.*

Keywords*:* *Inequalities Lowner partial ordering trace determinant*

1. Introduction

Theorem1 1 If $A \geq B \geq 0$, $AB = BA$, $f(x)$ is a monotonically increasing function on $(0, +\infty)$,then $f(A) \geq f(B)$.

Proof Because A, B are semi-positive matrices, and A, B can be exchanged, so there is an orthogonal matrix Q for

$$A = Q' \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} Q, \quad B = Q' \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix} Q$$

, and $\lambda_i \geq u_i, \forall i \in [1, n]$.

$$f(A) = Q' \begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{pmatrix} Q, \quad f(B) = Q' \begin{pmatrix} f(u_1) & & & \\ & f(u_2) & & \\ & & \ddots & \\ & & & f(u_n) \end{pmatrix} Q$$

, $f(x)$ is a monotonically increasing function on $(0, +\infty)$,so we have $f(\lambda_i) \geq f(u_i)$, $\forall i \in [1, n]$ .so

$f(A) \geq f(B)$.

With this proposition, we can get the following results:

Proposition 2 We assume $0 < A_k < I, k = 1, 2, ..., n$, and $A_i A_j = A_j A_i$, then $I - \sum_k A_k < (I + \sum_k A_k)^{-1}$.

Proof We know that

$$A_k = Q' \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix} Q$$

by $A_i A_j = A_j A_i$, Q is an orthogonal matrix ,and

$0 < a_i^k < 1$,then

$$I - \sum_k A_k = Q' \begin{pmatrix} 1 - \sum_i a_1^i & & & \\ & 1 - \sum_i a_2^i & & \\ & & \ddots & \\ & & & 1 - \sum_i a_n^i \end{pmatrix} Q$$

,

$$(I+\sum_k A_k)^{-1} = Q^t \begin{pmatrix} \frac{1}{1+\sum_i a_1^i} & & & \\ & \frac{1}{1+\sum_i a_2^i} & & \\ & & O & \\ & & & \frac{1}{1+\sum_i a_n^i} \end{pmatrix} Q$$,we get $1-\sum_i a_k^i < \frac{1}{1+\sum_i a_k^i}$ by Weierstrass inequality, so this proposition can be proved.

Corollary 3 We assume that $0 < A_k < I, k = 1, 2, \ldots, n$,and $A_i A_j = A_j A_i$.

Then $tr(I-\sum_k A_k) < tr(I+\sum_k A_k)^{-1}$

Corollary 4 We suppose that $0 < A_k < I, k = 1, 2, \ldots, n$,and $A_i A_j = A_j A_i$.

Then $\det(I-\sum_k A_k) < \det(I+\sum_k A_k)^{-1}$

Proposition 5 We suppose $A_1 \le A_2 \le \ldots \le A_n$ and $A_i A_j = A_j A_i$,

then $\sum_k (I+A_k)(I+A_{k+1})^{-1} \le nI + (I+A_1)^{-2}(\sum_k A_k - A_1)^2$,if and only $A_k = A_1$, the equation holds.

Corollary 6 We consider $A_1 \le A_2 \le \ldots \le A_n$,

and $A_i A_j = A_j A_i$, then $tr\sum_k (I+A_k)(I+A_{k+1})^{-1} \le n + tr[(I+A_1)^{-2}(\sum_k A_k - A_1)^2]$

Corollary 7 We assume $A_1 \le A_2 \le \ldots \le A_n$ and $A_i A_j = A_j A_i$,

then $\det[\sum_k (I+A_k)(I+A_{k+1})^{-1}] \le \det[nI + (I+A_1)^{-2}(\sum_k A_k - A_1)^2]$

Proposition 8 We suppose that $0 \le aI \le A_1 \le A_2, A_1A_2 = A_2A_1$.

Then $A_2^{\frac{1}{n}} - A_1^{\frac{1}{n}} \le (A_2 - aI)^{\frac{1}{n}} - (A_1 - aI)^{\frac{1}{n}} \le (A_2 - A_1)^{\frac{1}{n}}$.

Proposition 9 We suppose $A \ge B \ge 0$, $AB = BA$,then $(A^p - B^p) \le p(A-B)(A^{p-1} + B^{p-1})$, when $1 \le p < \infty$

Proposition 10 A, B are real symmetric positive definite matrices, $A \ge B$ and $AB = BA$, for every $n \in N$,we say

$(A-B)^{2n+1} \le 2^{2n}(A^{2n+1} - B^{2n+1})$

Proposition 11 $c > 0$ and A , B are real symmetric positive definite matrices, then $tr(A+B)^2 \le tr[(1+c)A^2+(1+\frac{1}{c})B^2]$.

Proposition 12 $1 \le p < \infty, 0 < t < 1$, A, B are real symmetric positive definite matrices, and $AB = BA$, then $(A+B)^p \le t^{1-p}A^p + (1-t)^{1-p}B^p$.

Proposition 13 We suppose $t_k \ge 0, \sum_{k=1}^{n} t_k = 1$, $A_1, \ldots, A_n$ are real symmetric positive definite matrices, and $A_iA_j = A_jA_i$,when $1 \le p < \infty$

$(\sum_{k=1}^{n} t_k A_k)^p \le (\sum_{k=1}^{n} t_k A_k^p)$ is established, when $0 < p < 1$, we get $(\sum_{k=1}^{n} t_k A_k)^p \ge (\sum_{k=1}^{n} t_k A_k^p)$.

Corollary 14 We assume that $t_k \ge 0, \sum_{k=1}^{n} t_k = 1, A_1, \ldots, A_n$ are real symmetric positive definite matrices, and $A_iA_j = A_jA_i$,when $1 \le p < \infty$, we have

$$tr(\sum_{k=1}^{n} t_k A_k)^p \le tr(\sum_{k=1}^{n} t_k A_k^p), \text{ when } 0 < p < 1, \text{ we get } tr(\sum_{k=1}^{n} t_k A_k)^p \ge tr(\sum_{k=1}^{n} t_k A_k^p).$$

Corollary 15 We suppose $t_k \ge 0, \sum_{k=1}^{n} t_k = 1, A_1, \ldots, A_n$ are real symmetric positive definite matrices, and $A_iA_j = A_jA_i$,when $1 \le p < \infty$, we have

$$\det(\sum_{k=1}^{n} t_k A_k)^p \le \det(\sum_{k=1}^{n} t_k A_k^p), \text{ when } 0 < p < 1, (\sum_{k=1}^{n} t_k A_k)^p \ge (\sum_{k=1}^{n} t_k A_k^p).$$

Proposition 16 $A_1, \ldots, A_n$ are real symmetric positive definite matrices, and $A_iA_j = A_jA_i$,when $p \ge 1$, we get that

$$\sum_k A_k{}^p \le (\sum_k A_k)^p \le n^{p-1} \sum_k A_k{}^p .$$

Proposition 17 $0 \le A \le I$, $p > 1$,then (1) $\frac{1}{2^{p-1}} I \le A^p + (I-A)^p \le I$, (2) $A^p(I-A)^p \le \frac{1}{4} I$.

Proof When $0 < x < 1$,we choose a function $f(x) = x^p + (1-x)^p$,it is not difficult to get $f(x) \ge \frac{1}{2^{p-1}}$,and $f(0) = f(1) = 1$,so $f(x) \le 1$.

By the same way, we can prove $\frac{1}{2^{p-1}} I \le A^p + (I-A)^p \le I$.

Similarly, we get $A^p(I-A)^p \le \frac{1}{4} I$.

Proposition 18 $A > 0$,when $a > 1$,we have $(I+A)^a > I + aA - \frac{1}{2}a(1-a)[A(I+A)^{-1}]^2$.

Proof We notice that when $x > 0$ and $a > 1$, $(I+x)^a > I + ax - \frac{1}{2}a(1-a)(\frac{x}{1+x})^2$ is established. Then we can prove this Proposition.

Corollary 19 $A > 0$, n is a positive integer, then

(1) $tr(I+A)^n > n + tr[nA - \frac{1}{2}n(1-n)[A(I+A)^{-1}]^2]$

(2) $\det(I+A)^n > \det[I + nA - \frac{1}{2}n(1-n)[A(I+A)^{-1}]^2]$

Proposition 20 $0 < a < 1$, $A + I \geq 0$,then $(I+A)^a \geq I + aA[I+(1-a)A]^{-1}$.

Proof when $0 < a < 1$, $x \geq -1$,we have $(1+x)^a \geq 1 + \dfrac{ax}{1+(1-a)x}$,so this proposition can be proved .

Note when $a > 1$ and $-I \leq A \leq \dfrac{1}{a-1}I$, $(I+A)^a \leq I + aA[I+(1-a)A]^{-1}$.

Proposition 21 $A + I \geq 0$, when $0 < a < 1$, $(I+A)^a \leq I + aA$, when $a < 0$ or $a > 1$,

$(I+A)^a \geq I + aA$

Corollary 22 $A + I \geq 0$,when $0 < a < 1$, $tr(I+A)^a \leq n + atrA$,when $a < 0$ or $a > 1$, $tr(I+A)^a \geq n + atrA$

Corollary 23 $A + I \geq 0$,when $0 < a < 1$, $\det(I+A)^a \leq \det(I+aA)$,when $a < 0$ or $a > 1$,we have $\det(I+A)^a \geq \det(I+aA)$

Corollary 24 $A \geq 0$, when $a < 0$ or $a > 1$, $\det(I+A)^a \geq \det(I+aA) \geq 1 + a^n \det A$.

Proposition 25 $A_1 \geq A_2 \geq .... \geq A_n \geq 0$ and $A_iA_j = A_jA_i$, $B_1, B_2, ..., B_n$ are real symmetric matrices, if $tr\sum_{i=1}^{k} A_i \leq tr\sum_{i=1}^{k} B_i$, $1 \leq k \leq n$,then $tr\sum_{i=1}^{n} A_i^2 \leq tr\sum_{i=1}^{n} B_i^2$.

REFERENCES

[1] Akhmedova V, Zabrodin A. Dispersionless DKP hierarchy and elliptic Löwner equation[J]. Journal of Physics A Mathematical & Theoretical, 2014, 47(47).

[2] Van Barel M, Vavrin Z. Inversion of a block Löwner matrix[J]. Journal of Computational & Applied Mathematics, 1996, 69(2):261-284.

[3] Nienhuis B, Kager W. Stochastic Löwner Evolution and the Scaling Limit of Critical Models[J]. Lecture Notes in Physics, 2009, 775:425-467.

[4] Nienhuis B, Kager W. Stochastic Löwner Evolution and the Scaling Limit of Critical Models[M]// Polygons, Polyominoes and Polycubes. 2009.

[5] Rost K, Vavřín Z. Recursive solution of Löwner-Vandermonde systems of equations. I ☆[J]. Journal of Computational & Applied Mathematics, 1996, 114(2):319-331.

[6] Uchiyama M. Operator Monotone Functions which Are Defined Implicitly and Operator Inequalities[J]. Journal of Functional Analysis, 2000, 175(2):330-347.

Citation: *Feng Zhang & Jinli Xu, (2019). Some Inequalities on Semi-Positive Definite Matrices. International Journal of Scientific and Innovative Mathematical Research (IJSIMR), 7(4), pp.28-31. http://dx.doi.org/10.20431/ 2347-3142 .0704005*