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Strong Insertion of a Contra-Continuous Function between Two Comparable Contra-Precontinuous (Contra-Semi−Continuous) Functions

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Abstract: *Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.*

1. Introduction

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [4]. A subset A of a topological space (X,τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [20], while the concept of a , locally dense, set was introduced by H.H. Corson and E. Michael [4].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [17]. A subset A of a topological space (X,τ) is called *semiopen* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [19].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [25] if the preimage of every open subset of R belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [5 , 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contracontinuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 24].

Hence, a real-valued function f defined on a topological space X is called *contra-continuous* (resp. *contra-semi−continuous*, *contra-precontinuous*) if the preimage of every open subset of R is closed (resp. *semi*−closed, preclosed) in X[6].

Results of Katˇetov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable realvalued functions on such topological spaces that $\Lambda-$sets or kernel of sets are open [19].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a *cc−property* provided that any constant function has property P and provided that the sum of a function with property P and any contracontinuous function also has property P. If P_1 and P_2 are cc−properties, the following terminology is used:(i) A space X has the *weak cc−insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g \leq f$,g has property P_1 and f has property P_2, then there exists a contracontinuous function h such that $g \leq h \leq f$.(ii) A space X has the *strong cc−insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g \leq f$,g has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose Λ−sets or kernel of sets are open, is given a sufficient condition for the weak cc−insertion property.

Also for a space with the weak cc−insertion property, we give necessary and sufficient conditions for the space to have the strong cc−insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

The abbreviations *cc*, *cpc* and *csc* are used for contra-continuous, contraprecontinuous and contra-*semi*−continuous, respectively.

Definition 2.1. Let A be a subset of a topological space (X,τ). We define the subsets A^{Λ} and A^{V} as follows:

$A^{\Lambda} = \cap\{O : O \supseteq A, O \in (X,\tau)\}$ and $A^{V} = \cup\{F : F \subseteq A, F^{c} \in (X,\tau)\}$.

In [7, 18, 23], A^{Λ} is called the *kernel* of A.

The family of all preopen, preclosed, *semi*−open and *semi*−closed will be denoted by $pO(X,\tau)$, $pC(X,\tau)$, $sO(X,\tau)$ and $sC(X,\tau)$, respectively.

We define the subsets $p(A^{\Lambda})$,$p(A^{V})$,$s(A^{\Lambda})$ and $s(A^{V})$ as follows: $p(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in pO(X,\tau)\}$, $p(A^{V}) = \cup\{F : F \subseteq A, F \in pC(X,\tau)\}$, $s(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in sO(X,\tau)\}$ and $s(A^{V}) = \cup\{F : F \subseteq A, F \in sC(X,\tau)\}$. $p(A^{\Lambda})$ (resp. $s(A^{\Lambda})$) is called the *prekernel* (resp. *semi − kernel*) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

- If $A_i \rho B_j$ for any $i \in\{1,...,m\}$ and for any $j \in\{1,...,n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in\{1,...,m\}$ and any $j \in\{1,...,n\}$.
- If $A \subseteq B$, then $A \bar{\rho} B$.
- If $A \rho B$, then $A^{\Lambda} \subseteq B$ and $A \subseteq B^{V}$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f,\ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1) \rho A(g,t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$. **Proof.** Theorem 2.1, of [22].

Theorem 2.2. Let P_1 and P_2 be cc−property and X be a space that satisfies the weak cc−insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g \leq f$,g has property P_1 and f has property P_2. The space X has the strong cc−insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g,2^{-n})$ and there exists a sequence $\{H_n\}$ of subsets of X such that (i) for each n,H_n and $A(f-g,2^{-n})$ are completely separated by contra-continuous functions, and ($\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ii) .

Proof. Theorem 3.1, of [21].

Theorem 2.3. Let P_1 and P_2 be cc−properties and assume that the space X satisfied the weak cc−insertion property for (P_1,P_2). The space X satisfies the strong cc−insertion property for (P_1,P_2) if and only if X satisfies the strong cc−insertion property for (P_1,cc) and for (cc,P_2). **Proof.** Theorem 3.2, of [21].

3. Applications

Before stating the consequences of theorems 2.1, 2.2, and 2.3 we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. If for each pair of disjoint preopen (resp. *semi*−open) sets G_1,G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X has the weak cc−insertion property for

(cpc,cpc) (resp. (csc,csc)).

Proof. Corollary 3.1, of [22].

Corollary 3.2. If for each pair of disjoint preopen (resp. *semi*−open) sets

G_1,G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then every contra-precontinuous (resp. contra-*semi*−continuous) function is contra-continuous.

Proof. Corollary 3.2, of [22].

Corollary 3.3. If for each pair of disjoint preopen (resp. *semi*−open) sets G_1,G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X has the cc−insertion property for (cpc,cpc)

(resp. (csc,csc)).

Proof. Corollary 3.3, of [22].

Corollary 3.4. If for each pair of disjoint subsets G_1,G_2 of X , such that G_1 is preopen and G_2 is *semi*−open, there exist closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X have the weak cc−insertion property for (cpc,csc) and (csc,cpc).

Proof. Corollary 3.4, of [22].

Before stating consequences of Theorem 2.2, 2.3 we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space X are equivalent:

- For each pair of disjoint subsets G_1,G_2 of X, such that G_1 is preopen and G_2 is *semi*−open, there exist closed subsets F_1,F_2 of X such that $G_1 \subseteq$

  $F_1,G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$.

- If G is a *semi*−open (resp. preopen) subset of X which is contained in a preclosed (resp. *semi*−closed) subset F of X, then there exists a closed subset H of X such that $G \subseteq H \subseteq H^{\wedge} \subseteq F$.

  Proof. Lemma 3.1, of [22].

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1,G_2 of X, where G_1 is preopen and G_2 is *semi*−open, can be separated by closed subsets of X then there exists a contra-continuous function $h : X \to [0,1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$. **Proof.** Lemma 3.2, of [22].

Lemma 3.3. Suppose that X is a topological space. If each pair of disjoint subsets G_1,G_2 of X, where G_1 is preopen and G_2 is *semi*−open, can separate by closed subsets of X, and G_1 (resp. G_2) is a closed subsets of X, then there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$).

Proof. Suppose that G_1 (resp. G_2) is a closed subset of X. By Lemma 3.2 , there exists a contra-continuous function $h : X \to [0,1]$ such that, $h(G_1) = \{0\}$ (resp. $h(G_2) = \{0\}$) and $h(X \setminus G_1) = \{1\}$ (resp. $h(X \setminus G_2) = \{1\}$). Hence, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and since $G_2 \subseteq X \setminus G_1$ (resp. $G_1 \subseteq X \setminus G_2$), therefore $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$).

Lemma 3.4. Suppose that X is a topological space such that every two disjoint *semi*−open and preopen subsets of X can be separated by closed subsets of X. The following conditions are equivalent:

- For every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen and G_2 is *semi*−open, there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h^{-1}(1) = G_2$ (resp. $h^{-1}(1) = G_1$).
- Every preopen (resp. *semi*−open) subset of X is a closed subsets of X.
- Every preclosed (resp. *semi*−closed) subset of X is an open subsets of X.

Proof. (i) ⇒ (ii) Suppose that G is a preopen (resp. *semi*−open) subset of X. Since ∅ is a *semi*−open (resp. preopen) subset of X, by (i) there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G$. Set $F_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, F_n is a closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G$.

(ii) ⇒ (i) Suppose that G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is *semi*−open. By Lemma 3.3, there exists a contracontinuous function $f : X \to [0,1]$ such that, $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two open subsets of X and

$(G \cup F) \cap G_2 = \emptyset$. By Lemma 3.3, there exists a contra-continuous function

$g : X \to [\frac{1}{2},1]$ such that, $g^{-1}(1) = G_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then h is welldefined and a contra-continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to [0,1]. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G2$.

(ii) ⇔ (iii) By De Morgan law and noting that the complement of every open subset of X is a closed subset of X and complement of every closed subset of X is an open subset of X, the equivalence is hold.

Corollary 3.5. If for every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen (resp. *semi*−open) and G_2 is *semi*−open (resp. preopen), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$ then X has the strong *cc*−insertion property for (*cpc*,*csc*) (resp. (*csc*,*cpc*)).

Proof. Since for every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen (resp. *semi*−open) and G_2 is *semi*−open (resp. preopen), there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$.

Then F_1 and F_2 are two disjoint closed subsets of X that contain G_1 and G_2, respectively. Hence by Corollary 3.4, X has the weak *cc*−insertion property for (*cpc*,*csc*) and (*csc*,*cpc*). Now, assume that g and f are functions on X such that $g \leq f$,g is *cpc* (resp. *csc*) and f is *cc*. Since $f - g$ is *cpc* (resp. *csc*), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a preopen (resp. *semi*−open) subset of X. Now setting $H_n = \{x \in X : (f - g)(x) > 2^{-n}\}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, H_n is an open subset of X and we have $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}$,H_n and $A(f - g, 2^{-n})$ are disjoint subsets of X. By Lemma 3.2, H_n and $A(f - g, 2^{-n})$ can be completely separated by contra-continuous functions. Hence by Theorem 2.2, X has the strong *cc*−insertion property for (*cpc*,*cc*) (resp. (*csc*,*cc*)).

By an analogous argument, we can prove that X has the strong *cc*−insertion property for (*cc*,*csc*) (resp. (*cc*,*cpc*)). Hence, by Theorem 2.3, X has the strong *cc*−insertion property for (*cpc*,*csc*) (resp. (*csc*,*cpc*)).

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