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Insertion of a Contra-Continuous Function between two Comparable Contra-Precontinuous (Contra-Semi−Continuous) Functions

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Abstract: *A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.*

Keywords: *Insertion, Strong binary relation, Semi-open set, Preopen set, Contracontinuous function, Lower cut set.*

1. Introduction

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [4]. A subset A of a topological space (X,τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [20], while the concept of a , locally dense, set was introduced by H.H. Corson and E. Michael [4].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [17]. A subset A of a topological space (X,τ) is called *semiopen* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [19].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [24] if the preimage of every open subset of R belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [5 , 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contracontinuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 23].

Hence, a real-valued function f defined on a topological space X is called *contra-continuous* (resp. *contra-semi−continuous* , *contra-precontinuous*) if the preimage of every open subset of R is closed (resp. *semi*−closed , preclosed) in X[6].

Results of Katˇetov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable realvalued functions on such topological spaces that $\Lambda-$sets or kernel of sets are open [19].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a *cc−property* provided that any constant function has property P and provided that the sum of a function with property P and any contracontinuous function also has property P. If P_1 and P_2 are cc−properties, the following terminology is used:(i) A space X has the *weak cc−insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g \leq f$,g has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$.(ii) A space X has the *cc−insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g < f$,g has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g < h < f$.(iii) A space X has the *weakly cc−insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g < f$,g has property P_1 , f has property P_2 and $f - g$ has property P_2, then there exists a contra-continuous function h such that $g < h < f$.

In this paper, for a topological space whose Λ−sets or kernel of sets are open, is given a sufficient condition for the weak cc−insertion property. Also for a space with the weak cc−insertion property, we give a necessary and sufficient condition for the space to have the cc−insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated. **Definition 2.1.** Let A be a subset of a topological space (X,τ). We define the subsets A^{Λ} and A^{V} as follows:

$A^{\Lambda} = \cap\{O : O \supseteq A, O \in (X,\tau)\}$ and $A^{V} = \cup\{F : F \subseteq A, F^{c} \in (X,\tau)\}$.

In [7, 18, 22], A^{Λ} is called the *kernel* of A.

The family of all preopen, preclosed, *semi*−open and *semi*−closed will be denoted by $pO(X,\tau)$, $pC(X,\tau)$, $sO(X,\tau)$ and $sC(X,\tau)$, respectively.

We define the subsets $p(A^{\Lambda})$,$p(A^{V})$,$s(A^{\Lambda})$ and $s(A^{V})$ as follows: $p(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in pO(X,\tau)\}$, $p(A^{V}) = \cup\{F : F \subseteq A, F \in pC(X,\tau)\}$, $s(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in sO(X,\tau)\}$ and $s(A^{V}) = \cup\{F : F \subseteq A, F \in sC(X,\tau)\}$. $p(A^{\Lambda})$ (resp. $s(A^{\Lambda})$) is called the *prekernel* (resp. *semi − kernel*) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

- If $A_i \rho B_j$ for any $i \in \{1,\dots,m\}$ and for any $j \in \{1,\dots,n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1,\dots,m\}$ and any $j \in \{1,\dots,n\}$.
- If $A \subseteq B$, then $A \bar{\rho} B$.
- If $A \rho B$, then $A^{\Lambda} \subseteq B$ and $A \subseteq B^{V}$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if {$x \in X : f(x) <$ `}⊆ $A(f,$`) ⊆{$x \in X : f(x) \leq$ `} for a real number `, then $A(f,$`) is called a *lower indefinite cut set* in the domain of f at the level

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1) \rho A(g,t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers Q into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [15] it follows that there exists a function H mapping Q into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$. For any x in X, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t^0)$ for any $t^0 > t$; since x is in $G(t^0) = A(g,t^0)$ implies that $g(x) \leq t^0$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t^0)$ for any $t^0 < t$; since x is not in $F(t^0) = A(f,t^0)$ implies that $f(x) > t^0$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is closed in X, i.e., h is a contra-continuous function on X.

The above proof used the technique of theorem 1 in [14].

Theorem 2.2. Let P_1 and P_2 be $cc-$property and X be a space that satisfies the weak $cc-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g < f$, g has property P_1 and f has property P_2.The space X has the $cc-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g,3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each n, $X \setminus D_n$ and $A(f-g,3^{-n+1})$ are completely separated by contra-continuous functions.

Proof. Theorem 2.1 of [21].

3. Applications

The abbreviations cpc and csc are used for contra-precontinuous and contra*semi*−continuous, respectively.

Before stating the consequences of theorems 2.1, 2.2, we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. If for each pair of disjoint preopen (resp. *semi*−open) sets

G_1,G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X has the weak $cc-$insertion property for

(cpc,cpc) (resp. (csc,csc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $p(A^\Lambda) \subseteq p(B^V)$ (resp. $s(A^\Lambda) \subseteq s(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2)$;

since $\{x \in X : f(x) \leq t_1\}$ is a preopen (resp. *semi*−open) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. *semi*−closed) set, it follows that $p(A(f,t_1)^\Lambda) \subseteq p(A(g,t_2)^V)$ (resp. $s(A(f,t_1)^\Lambda) \subseteq s(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1.

Corollary 3.2. If for each pair of disjoint preopen (resp. *semi*−open) sets

G_1,G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then every contra-precontinuous (resp. contra-*semi*−continuous) function is contra-continuous.

Proof. Let f be a real-valued contra-precontinuous (resp. contra-*semi*−continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contracontinuous function h such that $g = h = f$.

Corollary 3.3. If for each pair of disjoint preopen (resp. *semi*−open) sets

G_1,G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X has the $cc-$insertion property for (cpc,cpc)

(resp. (csc,csc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are cpc (resp. csc), and $g < f$. Set $h = (f + g)/2$, thus $g < h < f$, and by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function.

Corollary 3.4. If for each pair of disjoint subsets G_1,G_2 of X , such that G_1 is preopen and G_2 is *semi*−open, there exist closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X have the weak cc−insertion property for (cpc,csc) and (csc,cpc).

Proof. Let g and f be real-valued functions defined on X, such that g is cpc (resp. csc) and f is csc (resp. cpc), with $g \leq f$.If a binary relation ρ is defined by $A \ \rho \ B$ in case $s(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $p(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then

$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2)$;

since $\{x \in X : f(x) \leq t_1\}$ is a *semi*−open (resp. preopen) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. *semi*−closed) set, it follows that $s(A(f,t_1)^{\Lambda}) \subseteq p(A(g,t_2)^{V})$ (resp. $p(A(f,t_1)^{\Lambda}) \subseteq s(A(g,t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f,t_1) \ \rho \ A(g,t_2)$. The proof follows from Theorem 2.1.

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space X are equivalent:

- For each pair of disjoint subsets G_1,G_2 of X, such that G_1 is preopen and G_2 is *semi*−open, there exist closed subsets F_1,F_2 of X such that $G_1 \subseteq F_1$,$G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$.
- If G is a *semi*−open (resp. preopen) subset of X which is contained in a preclosed (resp. *semi*−closed) subset F of X, then there exists a closed subset H of X such that $G \subseteq H \subseteq H^{\Lambda} \subseteq F$.

Proof. (i) ⇒ (ii) Suppose that $G \subseteq F$, where G and F are *semi*−open

(resp. preopen) and preclosed (resp. *semi*−closed) subsets of X, respectively. Hence, F^c is a preopen (resp. *semi*−open) and $G \cap F^c = \emptyset$.

By (i) there exists two disjoint closed subsets F_1,F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an open subset containing F_1, we conclude that $F_1^{\Lambda} \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^{\Lambda} \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) ⇒ (i) Suppose that G_1,G_2 are two disjoint subsets of X, such that G_1 is preopen and G_2 is *semi*−open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is a preclosed subset of X. Hence by (ii) there exists a closed set H such that $G_2 \subseteq H \subseteq H^{\Lambda} \subseteq G_1^c$.

But $H \subseteq H^{\Lambda} \Rightarrow H \cap (H^{\Lambda})^c = \emptyset$

and

$$H^{\Lambda} \subseteq G_1^c \Rightarrow G_1 \subseteq (H^{\Lambda})^c.$$

Furthermore, $(H^{\Lambda})^c$ is a closed subset of X. Hence $G_2 \subseteq H$,$G_1 \subseteq (H^{\Lambda})^c$ and $H \cap (H^{\Lambda})^c = \emptyset$. This means that condition (i) holds.

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1,G_2 of X, where G_1 is preopen and G_2 is *semi*−open, can be separated by closed subsets of X then there exists a contra-continuous function $h : X \to [0,1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is *semi*−open. Since $G_1 \cap G_2 = \emptyset$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is a preclosed subset of X containing the *semi*−open subset G_2 of X,by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq G_1^c.$$

Note that $H_{1/2}$ is also a preclosed subset of X and contains G_2, and G_1^c is a preclosed subset of X and contains the *semi*−open subset $H_{1/2}^{\Lambda}$ of X. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^{\Lambda} \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq H_{3/4} \subseteq H_{3/4}^{\Lambda} \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets H_t with the property that if $t_1,t_2 \in D$ and $t_1 < t_2$, then $H_{t1} \subseteq H_{t2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for x 6∈ G_1 and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into $[0,1]$. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contracontinuous function on X. For every $\alpha \in$R, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(H_t^{\Lambda})^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently h is a contra-continuous function.

Lemma 3.3. Suppose that X is a topological space such that every two disjoint *semi*−open and preopen subsets of X can be separated by closed subsets of X. The following conditions are equivalent:

- Every countable convering of semi−closed (resp. preclosed) subsets of X has a refinement consisting of preclosed (resp. semi−closed) subsets of X such that for every x ∈ X, there exists a closed subset of X containing x such that it intersects only finitely many members of the refinement.
- Corresponding to every decreasing sequence {Gn} of semi−open (resp. preopen) subsets of X with empty intersection there exists a decreasing sequence {Fn} of preclosed (resp. semi−closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every n ∈N,Gn ⊆ Fn.

Proof. (i) ⇒ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of *semi*−open (resp. preopen) subsets of X with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of *semi*−closed (resp. preclosed) subsets of X . By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in$N$\}$ such that every V_n is a closed subset of X and $V_n^{\Lambda} \subseteq G_n^c$. By setting $F_n = (V_n^{\Lambda})^c$, we obtain a decreasing sequence of closed subsets of X with the required properties.

(ii) ⇒ (i) Now if $\{H_n : n \in$N$\}$ is a countable covering of *semi*−closed (resp. preclosed) subsets of X, we set $n \in \mathbb{N}, G_n = (\bigcup_{i=1}^{n} H_i)^c$ for . Then $\{G_n\}$ is a decreasing sequence of *semi*−open (resp. preopen) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of preclosed (resp. *semi*−closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in$N,$G_n \subseteq F_n$.Now we define the subsets W_n of X in the following manner:

W_1 is a closed subset of X such that $F_1^c \subseteq W_1$ and $W_1^{\Lambda} \cap G_1 = \varnothing$.

W_2 is a closed subset of X such that $W_1^{\Lambda} \cup F_2^c \subseteq W_2$ and $W_2^{\Lambda} \cap G_2 = \varnothing$, and so on. (By Lemma 3.1, W_n exists).

Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in$N$\}$ is a

covering for X consisting of closed sets. Moreover, we have

(i) $Wn\Lambda \subseteq Wn+1$

(ii) $F_n^c \subseteq W_n$

(iii) $Wn \subseteq Sni=1\ Hi$.

Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus W_{n-1}^{\Lambda}$.

Then since $W_{n-1}^{\Lambda} \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of closed sets and covers X. Furthermore, $S_i \cap S_j 6= \emptyset$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$S_1 \cap H_1$, $S1 \cap H2$

$S_2 \cap H_1$, $S_2 \cap H_2$, $S2 \cap H3$

$S_3 \cap H_1$, $S_3 \cap H_2$, $S_3 \cap H_3$, $S3 \cap H4$

...

$S_i \cap H_1$, $S_i \cap H_2$, $S_i \cap H_3$, $S_i \cap H_4$, $\cdots$, $Si \cap Hi+1$

These sets are closed sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a closed set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1,...,i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are closed sets, and for every point in X we can find a closed set containing the point that intersects only finitely many elements of that refinement.

Corollary 3.5. If every two disjoint *semi*−open and preopen subsets of X can be separated by closed subsets of X, and in addition, every countable covering of *semi*−closed (resp. preclosed) subsets of X has a refinement that consists of preclosed (resp. *semi*−closed) subsets of X such that for every point of X we can find a closed subset containing that point such that it intersects only a finite number of refining members then X has the weakly *cc*−insertion property for (*cpc*,*csc*) (resp. (*csc*,*cpc*)).

Proof. Since every two disjoint *semi*−open and preopen sets can be separated by closed subsets of X, therefore by Corollary 3.4, X has the weak *cc*−insertion property for (*cpc*,*csc*) and (*csc*,*cpc*). Now suppose that f and g are real-valued functions on X with $g < f$, such that g is *cpc* (resp. *csc*) , f is *csc* (resp. *cpc*) and $f-g$ is *csc* (resp. *cpc*). For every $n \in \mathbb{N}$, set

$A(f-g,3^{-n+1}) = \{x \in X : (f-g)(x) \leq 3^{-n+1}\}$.

Since $f-g$ is *csc* (resp. *cpc*), hence $A(f-g,3^{-n+1})$ is a *semi*−open (resp. preopen) subset of X. Consequently, $\{A(f-g,3^{-n+1})\}$ is a decreasing sequence of *semi*−open (resp. preopen) subsets of X and furthermore since

$0 < f-g$, it follows that $\bigcap_{n=1}^{\infty} A(f-g,3^{-n+1}) = \varnothing$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of preclosed (resp. *semi*−closed) subsets of X such that $A(f-g,3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma

3.2, the pair $A(f-g,3^{-n+1})$ and $X \setminus D_n$ of *semi*−open (resp. preopen) and preopen (resp. *semi*−open) subsets of X can be completely separated by contra-continuous functions. Hence by Theorem 2.2, there exists a contracontinuous function h defined on X such that $g < h < f$, i.e., X has the weakly *cc*−insertion property for (*cpc*,*csc*) (resp. (*csc*,*cpc*)).

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