International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 7, Issue 1, 2019, PP 1-3
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0701001
www.arcjournals.org

Linear Transformations on Two Dimensions Delays Differential Equations Preserving Dynamics

Lin Deng, Feng Zhang, Jinli Xu

Department of mathematics, Northeast Forestry University, Harbin, China

Corresponding Author:** ***Lin Deng, *Department of mathematics, Northeast Forestry University, Harbin, China*

Abstract: *Let* R *be the real field. We consider the two dimension systems:* $\dot{x}(t)=Ax(t)+Bx(t-\tau)$ *where* $A,B\in R^{2\times 2}$, $\tau>0$. *The characteristic polynomial of above system is* $\det\left(\lambda I-A-Be^{-\lambda\tau}\right)$, *we determine the form of linear map* $\phi:R^{2\times 2}\to R^{2\times 2}$ *preserving the characteristic polynomial.*

Keywords: *delays differential equations, linear preserver problem, linear map*

1. Introduction

Let R be the real field. We consider the two dimension systems:

$$\dot{x}(t)=Ax(t)+Bx(t-\tau) \quad (1)$$

where $A,B\in R^{2\times 2}$, $\tau>0$. The characteristic polynomial of (1) is

$$\det\left(\lambda I-A-Be^{-\lambda\tau}\right)$$

In this paper, we determine the form of linear map $\phi:R^{2\times 2}\to R^{2\times 2}$ preserving the characteristic equation of (1).

Theorem 1. ([1,2,3]) Suppose $\phi:R^{n\times n}\to R^{n\times n}$ is a linear map. Then

$$\det(\lambda I-A)=\det(\lambda I-\phi(A)) \text{ for all } A\in R^{n\times n}$$

if and only if ϕ is of the form

$$X\mapsto PXP^{-1}, \text{ or } X\mapsto PX^{T}P^{-1}, \forall X\in R^{n\times n}$$

where P is a nonsingular matrix.

Theorem 2. Suppose $\phi:R^{n\times n}\to R^{n\times n}$ is a linear map. Then

$$\det\left(\lambda I-Be^{-\lambda\tau}\right)=\det\left(\lambda I-\phi(B)e^{-\lambda\tau}\right) \text{ for all } B\in R^{n\times n}$$

if and only if ϕ is of the form

$$X\mapsto PXP^{-1}, \text{ or } X\mapsto PX^{T}P^{-1}, \forall X\in R^{n\times n}$$

where P is a nonsingular matrix.

Proof. Let $E_r(X)$ is the sum of all principal $r\times r$ sub determinants of X. It is easy to see

$$\det\left(\lambda I-Be^{-\lambda\tau}\right)=\Sigma_r(-1)^r E_r(B)\lambda^{n-r}e^{-r\lambda\tau}.$$

By $\det\left(\lambda I-Be^{-\lambda\tau}\right)=\det\left(\lambda I-\phi(B)e^{-\lambda\tau}\right)$, we obtain $E_r(B)=E_r(\phi(B))$. Hence

$$\det(\lambda I-B)=\Sigma_r(-1)^r E_r(B)\lambda^{n-r}=\Sigma_r(-1)^r E_r(\phi(B))\lambda^{n-r}=\det(\lambda I-\phi(B)),$$

That is ϕ preserving ordinary characteristic polynomial.

Theorem 3. Suppose $\phi: R^{n\times n} \to R^{n\times n}$ is a linear map and $B \in R^{n\times n}$ is nonzero matrix. Then

$$\det\left(\lambda I - A - Be^{-\lambda\tau}\right) = \det\left(\lambda I - \phi(A) - \phi(B)e^{-\lambda\tau}\right) \text{ for all } A \in R^{n\times n}$$

if and only if ϕ is of the form

$X \mapsto PXP^{-1}$, or $X \mapsto PX^T P^{-1}, \forall X \in R^{n\times n}$

where P is a nonsingular matrix.

Proof. Setting $A = 0$, we obtain

$\det\left(\lambda I - Be^{-\lambda\tau}\right) = \det\left(\lambda I - \phi(B)e^{-\lambda\tau}\right)$.

Hence, B and $\phi(B)$ have the same characteristic polynomial, so are the eigenvalues. Without loss of generality, we may assume that B and $\phi(B)$ are already in their canonical form.

We next assume $n = 2$.

Case 1. B has mutually different eigenvalues. In this case, B and $\phi(B)$ has the common canonical form, say $B = \phi(B) = b_1 \oplus b_2$. We assume $\phi(E_{11}) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ in each determine. By

$$\det\begin{bmatrix} \lambda - b_1 e^{-\lambda\tau} - 1 & 0 \\ 0 & \lambda - b_2 e^{-\lambda\tau} \end{bmatrix} = \det\begin{bmatrix} \lambda - b_1 e^{-\lambda\tau} - x_{11} & -x_{12} \\ -x_{21} & \lambda - b_2 e^{-\lambda\tau} - x_{22} \end{bmatrix}$$

that is

$$\begin{aligned}\left(\lambda - b_1 e^{-\lambda\tau} - 1\right)\left(\lambda - b_2 e^{-\lambda\tau}\right) &= \lambda^2 - \lambda - (b_1 + b_2)\lambda e^{-\lambda\tau} + b_2 e^{-\lambda\tau} + b_1 b_2 e^{-2\lambda\tau} \\ &= \lambda^2 - \left(x_{11} + x_{22}\right)\lambda - \left(b_1 + b_2\right)\lambda e^{-\lambda\tau} + \left(x_{22} b_1 + x_{11} b_2\right) e^{-\lambda\tau} \\ &\quad + b_1 b_2 e^{-2\lambda\tau} + x_{11}x_{22} - x_{12}x_{21}\end{aligned}$$

Hence,

$$(x_{11} + x_{22} - 1)\lambda + (x_{22}b_1 + x_{11}b_2 - b_2)e^{-\lambda\tau} + (x_{11}x_{22} - x_{12}x_{21}) = 0$$

We have $x_{11} = 1$, $x_{22} = 0$ and $x_{12}x_{21} = 0$. Similarly, we can obtain $\phi(E_{22}) = \begin{bmatrix} 0 & y_{12} \\ y_{21} & 1 \end{bmatrix}$, with $y_{12}y_{21} = 0$. $\phi(E_{12}) = \begin{bmatrix} 0 & z_{12} \\ z_{21} & 0 \end{bmatrix}$, with $z_{12}z_{21} = 0$, $\phi(E_{21}) = \begin{bmatrix} 0 & w_{12} \\ w_{21} & 0 \end{bmatrix}$, with $w_{12}w_{21} = 0$. We assume $z_{12} \neq 0$, then $z_{21} = 0$. It is easy to see $w_{12} = 0$, and $w_{21} \neq 0$, and $z_{12}w_{21} = 1$. Thus, we can obtain $x_{12} = 0$, $x_{21} = 0$, $y_{12} = 0$, $y_{21} = 0$, hence, $\phi(E_{11}) = E_{11}$, and $\phi(E_{22}) = E_{22}$. Let $P = 1 \oplus z_{12}^{-1}$, then $\phi(X) = PXP^{-1}$.

Case 2. $B = \mu I_2$. Then $\phi(B) = \mu I_2$, or $\phi(B) = \mu I_2 + E_{12}$.

Subcase I. $B = \phi(B) = \mu I_2$. Similar as Case 1, we can see $\det\phi(E_{11}) = 0$, and $tr\phi(E_{11}) = 1$, without loss of generality, we can assume $\phi(E_{11}) = E_{11}$. By $B = \phi(B) = \mu I_2$, we can obtain $\phi(E_{22}) = E_{22}$. Using the similar method as Case 1, we see the result holds.

Subcase II. $B = \mu I_2$ and $\phi(B) = \mu I_2 + E_{12}$. we will prove this case cannot appear.

Noting that

$$\det\begin{bmatrix} \lambda - \mu e^{-\lambda\tau} - a & -b \\ -c & \lambda - \mu e^{-\lambda\tau} - d \end{bmatrix} = \det\begin{bmatrix} \lambda - \mu e^{-\lambda\tau} - x & e^{-\lambda\tau} - y \\ -z & \lambda - \mu e^{-\lambda\tau} - u \end{bmatrix}$$

Hence $\lambda^2 - (a+d)\lambda + (a+d-2)\mu e^{-\lambda\tau} + \mu^2 e^{-2\lambda\tau} + ad - bc$ and

$\lambda^2 - (x+u)\lambda + (x+u-2)\mu e^{-\lambda\tau} + ze^{-\lambda\tau} + \mu^2 e^{-2\lambda\tau} + xu - yz$. Thus $a + d = x + u$,

$(a+d-2)\mu=(x+u-2)\mu+z,\ ad-bc=xu-yz.$

This implies $z=0$, i.e. $\phi(M_2)\subset T_2$ (the up triangle matrix set), and $\det X=\det\phi(X)$, which is a contradiction.

Case III. $B=\mu I_2+E_{12}$, similar to the above, and then we complete the proof.

References

[1] C. K. Li and S. Pierce, A survey on linear preserver problems, Amer. Math. Monthly, vol. 108, pp. 591-605, 2001.

[2] M. H. Chan, M. H. Lim and K. K. Tan, Linear preservers on matrices, Linear Algebra and its Applications, vol. 93, pp. 67-80, 1987.

[3] M. Marcus and B. N. Moyls, Linear transformtions on algebras of matrices, Canad. J. Math., vol. 61, pp. 61-66, 1959.

Citation: *Deng, L.et.al, (2019). Linear Transformations on Two Dimensions Delays Differential Equations Preserving Dynamics. International Journal of Scientific and Innovative Mathematical Research (IJSIMR), 7(1), pp.1-3. http://dx.doi.org/10.20431/2347-3142.0701001*