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Some Inequalities on Traces

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Abstract: *In this paper, we mainly prove the inequality of semi-positive matrices with respect to traces under special partial order and the generalization and proof of the identities of traces of higher-order matrices.*

Keywords: *Partial ordering Trace Semi-positive matrix*

1. Introduction

Trace is an important function in matrix algebra. It has many good properties. In this paper, we try to change the product of the matrix into the Hadamard product of the matrix, and get the correlation. Inequality and a special extension of the proof of the third-order special matrix trace given by Xia yin hong, and prove it. Finally, under the condition of the $L\ddot{o}wner$ partial ordering of the matrix, the inequalities of the second-order semi-positive matrices and the third-order semi-positive matrices are given, and the inequalities of the $n-th$ order semi-positive matrices in the $l\ddot{o}wner$ partial ordering are given.

Now let's discuss.

2. Some Inequalities on Semi-Positive Matrices' Traces

Lemma1. Let $Z_n=(E_{11},E_{22},...,E_{nn})$, then $A\circ B=Z_n(A\otimes B)Z_n^T$, $A\circ B$ represents the Hadamard product of the matrix, $A\otimes B$ represents the Kronecker product of the matrix.

Lemma2. Let $Z_n=(E_{11},E_{22},...,E_{nn})$, If X is an n^2 order real symmetric semi-positive definite array, then $Tr(Z_nXZ_n^T)\geq 0$.

Proof. If A and B are all real symmetric semi-positive matrices, we can know from Lemma 1 that $(A\circ B)^2=[Z_n(A\otimes B)Z_n^T][Z_n(A\otimes B)Z_n^T]$, $A^2\circ B^2=Z_n(A^2\otimes B^2)Z_n^T=Z_n(A\otimes B)(A\otimes B)Z_n^T$, and $A\otimes B$ is a Semi-positive matrix, $(I-Z_n^TZ_n)$ is a Semi-positive matrix, so $(A\otimes B)(A\otimes B)\geq(A\otimes B)Z_n^TZ_n(A\otimes B)$, and then by Lemma 2. $Tr(Z_n(A\otimes B)(I-Z_n^TZ_n)(A\otimes B)Z_n^T)\geq 0$. So there is $Tr(A\circ B)^2\leq TrA^2\circ B^2$.

Theorem 1. If A and B are real symmetric semi-positive matrices, we have

$Tr(A\circ B)^2\leq TrA^2\circ B^2$.

Lemma3. Assume A and B are real symmetry matrices.

(1) If A is positive or B is positive, the eigenvalues of AB are all real numbers.

(2) If A is positive, we have B is positive if and only if the eigenvalues of AB are all positive real numbers.

(3) If A and B are semi-positive such that the eigenvalues of AB are all non-negative real numbers.

Theorem2. If A and B are semi-positive matrices of the same order, m is a positive integer, such that $Tr(A+B)^m \geq TrA^m + TrB^m$.

Proof. By calculating and using Lemma 3,

$$Tr(A+B)^m - TrA^m - TrB^m \geq mTr[A^{m-1}B + A^{m-2}B^2 + A^{m-3}B^3 ... + AB^{m-1}]$$

and A^k, B^{m-k} are still semi-positive arrays, so $TrA^kB^{m-k} \geq 0$ .and

$Tr[A^{m-1}B + A^{m-2}B^2 + A^{m-3}B^3 ... + AB^{m-1}] = \sum_{i=1}^{m-1} Tr(A^{m-i}B^i) \geq 0$. This proves that

$Tr(A+B)^m \geq TrA^m + TrB^m$.

Corollary1. If $A_1, A_2, ..., A_n$ are n-th order semi-positive arrays, then

$$Tr(A_1 + A_2 + ... + A_n)^m \geq TrA_1^{\ m} + TrA_2^{\ m} + ... + TrA_n^{\ m}$$

The significance of Theorem 2 is that even if A and B do not satisfy the commutability, the trace of the positive integer power of $A+B$ and the trace of the positive integer power of A plus the positive integer power of B can be judged.

Now, a similarity is given to identity the third-order special matrix trace which is given by Xia Yin hong. The idea of Xia is if a 3x3 matrix has real eigenvalues $1, \lambda, \frac{1}{\lambda}$, the following equations hold:

(1) $(TrA)^2 = TrA^2 + 2TrA$,

(2) $(TrA)^3 + 2TrA^3 = 3(TrA)(TrA^2) + 6$

Theorem 3. Assume $A = \begin{pmatrix} \lambda & k_1 & \\ & \frac{1}{\lambda} & k_2 \\ & & M \end{pmatrix} (\lambda \in R)$, here $k_i = 0 or 1 (i = 1, 2)$, $(n-2) \times (n-2)$, the matrix M satisfies $\exists s \in N$, such that $(I - M)^s = 0$.

Therefore, we have:

(1) $(TrA)^2 = TrA^2 + 2(TrM)(TrA) - (TrM)^2 - TrM + 2$

(2) $(TrA)^3 = TrA^3 + 3TrA + TrM[2 + 3TrA^2 - 3TrM + 3(TrM)(TrA) - 2(TrM)^2]$

Proof . We only prove (1), and the proof of (2) is similar.

$$(TrA)^2 = (\lambda + \frac{1}{\lambda} + n - 2)^2 = \lambda^2 + \frac{1}{\lambda^2} + n - 2 - (n-2) + (n-2)^2 + 2(n-2)[\lambda + \frac{1}{\lambda} + (n-2) - (n-2)] + 2$$
$$= TrA^2 - TrM + (TrM)^2 + 2TrM[TrA - TrM] + 2 = TrA^2 + 2(TrM)(TrA) - (TrM)^2 - TrM + 2$$

It's easy to verify $(TrA)^2 = TrA^2 + 2TrA$ when $n = 3$, $M = 1$, in this condition, we also have $(TrA)^3 + 2TrA^3 = 3(TrA)(TrA^2) + 6$.

Corollary**1.** Let A is an n-order real symmetric positive definite matrix, and let B and C are the same-order real symmetric semi-positive definite matrices, and $B \geq C$ then

$\forall k>0$， $Tr((A+kB)^{-1}B)\geq Tr((A+kC)^{-1}C)$.

Proof. We notice that $Tr(A+kB)^{-1}B=\frac{1}{k}Tr(A+kB)^{-1}[(A+kB)-A]=\frac{1}{k}[n-Tr(A+kB)^{-1}A]$

$Tr(A+kC)^{-1}B=\frac{1}{k}Tr(A+kC)^{-1}[(A+kC)-A]=\frac{1}{k}[n-Tr(A+kC)^{-1}A]$

since $A+kB\geq A+kC$, then $(A+kC)^{-1}\geq(A+kB)^{-1}$, naturally, we can obtain $A^{\frac{1}{2}}(A+kC)^{-1}A^{\frac{1}{2}}\geq A^{\frac{1}{2}}(A+kB)^{-1}A^{\frac{1}{2}}$,so $Tr(A+kB)^{-1}A\leq Tr(A+kC)^{-1}A$,

thus, $Tr((A+kB)^{-1}B)\geq Tr((A+kC)^{-1}C)$.

Corollary 2. Let A is a real symmetric positive matrix, and B, C are two real symmetric semi-positive matrices, $B\geq C$, we have $Tr((A+B)^{-1}B)\geq Tr((A+C)^{-1}C)$.

Proposition2. If A,B are 2nd-order real symmetric positive definite matrices, $A<B$, then we state that $TrA^2<TrB^2$.

Proof. Since the trace is an invariant under similarity **,** we can set $A=\begin{pmatrix}a & b\\ b & c\end{pmatrix}, B=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$.we write $TrA^2=a^2+c^2+2b^2$, $TrB^2=\lambda_1{}^2+\lambda_2{}^2$. hence $A<B$, we have the inequalities $(\lambda_1-a)^2>b^2$， $(\lambda_2-c)^2>b^2$,then we

obtain $a^2+c^2+2b^2<a^2+c^2+(\lambda_1-a)^2+(\lambda_2-c)^2<2(a^2+c^2)+\lambda_1{}^2+\lambda_2{}^2-2a\lambda_1-2c\lambda_2$.

For $\lambda_1>a,\lambda_2>c$,

Then there is $2(a^2+c^2)+\lambda_1{}^2+\lambda_2{}^2-2a\lambda_1-2c\lambda_2<\lambda_1{}^2+\lambda_2{}^2$.

so we get $TrA^2<TrB^2$.

Proposition3. If A and B are 3rd-order real symmetric positive matrices, $A<B$, then $TrA^2<2TrB^2$.

Proof. We assume that $A=\begin{pmatrix}a & b & c\\ b & e & f\\ c & f & d\end{pmatrix}, B=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$,

then $TrA^2=a^2+e^2+d^2+2[b^2+c^2+f^2]$ $TrB^2=\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2$. we notice that $(\lambda_1-a)^2>b^2,(\lambda_1-a)^2>c^2$ $(\lambda_2-e)^2>b^2,(\lambda_2-e)^2>f^2$, $(\lambda_3-d)^2>c^2$,$(\lambda_3-d)^2>f^2$. Then we show that

$$a^2+e^2+d^2+2[b^2+c^2+f^2]\leq a^2+e^2+d^2+2[\lambda_1{}^2-2a\lambda_1+a^2+\lambda_2{}^2-2e\lambda_2+e^2+\lambda_3{}^2-2d\lambda_3+d^2]$$

$<a^2+e^2+d^2+2[\lambda_1{}^2-a^2+\lambda_2{}^2-e^2+\lambda_3{}^2-d^2]<2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)$, so we prove that $TrA^2<2TrB^2$.

In fact, for Proposition 2 and Proposition 3, the condition of the positive definite matrix can be weakened to a semi-positive definite matrix. This is Proposition 4.

Proposition 4 . If A,B,C,D are semi-positive matrices with $A<B$, $C<D$.Then there holds $TrAC\leq TrBD$.

Proof. It is not difficult to get that $TrAC\leq TrBC\leq TrBD$.

In fact, Proposition 2 and Proposition 3 can be regarded as the inference of Proposition 4. When A,B,C,D satisty $A=B,C=D$, the Proposition 2 and proposition 3 are established.

From Proposition 4, we can get the following conclusions. If A, B , C are three semi-positive definite matrices, $A\leq B$, $C\leq I$, then the following statement holds. $Tr(ACA)\leq TrB^2$ .In fact**,** $TrACA=TrCA^2\leq TrA^2\leq TrB^2$.

References

[1] J.K.Baksalary,R.Kala and K.Klaczynski ,Lin.Alg.Appl,54(1983),77-86

[2] J.K.Baksalary,F.Pukelsheim and G.P.H.Styan,Lin.Alg.Appl.119(1989),57-85

[3] J.K.Baksalary and F.Pukelsheim ,Lin.Alg.Appl.151(1991),135-141

[4] J.K.Baksalary and J.Hauke,Lin .Alg.Appl.127(1990).157-169

[5] J.K.Baksalary and S.K.Mitra ,Lin .Alg.Appl,149(1991).73-89

[6] J.K.Baksalary, B.Schipp and G.Trenkler ,Lin.Alg.Appl.160(1992),119-129

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