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Analysis of Some Selected Numerical Methods in Solving Second Degree Non-Linear Equations

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Abstract: *We derived and compared some iterative methods used in solving second degree nonlinear equations such as Muller, Chebyshev and Multi-Point Methods. We used equations from already published sources and simultaneously applied all three methods to provide the approximate solution of the equations which are second degree nonlinear equations. The results are presented side by side in tabular form to establish the rate of convergence and the root of the equations.*

Keywords: *Non-Linear System, Algebraic System, Multi-piont, Algorithm, Convergence*

1. Introduction

According to Louis et al. (2005), an equation is said to be nonlinear when it involves terms of degree higher than 1 in the unknown quantity. These terms may be polynomial or capable of being broken-down into Taylor Series of degrees higher than 1.

Over the years, we have been taught on how to solve equations using various algebraic methods. These methods include the substitution method and the elimination method. Other algebraic methods that can be executed include the quadratic formula and factorization. In Linear Algebra, we learned that solving systems of linear equations can be implemented by using row reduction as an algorithm. However, when these methods are not successful, we use the concept of numerical methods.

Numerical methods are used to approximate solutions of equations when exact solutions cannot be determined via algebraic methods. They construct successive approximations that converge to the exact solution of an equation or system of equations. In first degree nonlinear equations we focused on solving nonlinear equations involving only a single variable. We used methods such as Newton's method, the Secant method, and the Regula-falsi method. We also examined numerical methods such as the Runge-Kutta methods, which are used to solve initial-value problems for ordinary differential equations. However, these problems only focused on solving nonlinear equations with only one variable, rather than nonlinear equations with several variables.

A root of the algebraic equation $f(x) = 0$ can be obtained by using the iteration methods based on the second-degree equation. We approximate the given $f(x)$by a second-degree equation in the neighbourhood of the root of $f(x) = 0$.

Nonlinear equations are generally solved using iterative methods. Iteration methods are based on the idea of successive approximations, starting with one or more initial approximations to the root. As defined by Barret et al. (1994).

The second-degree nonlinear equation defined over a variable x is

$$f(x) = a_0x^2 + a_1x + a_2 = 0 \ldots \qquad (1.1)$$

with $a_0 \neq 0$, a_0, $a_1$and a_2 are arbitrary parameters to be determined.

According to Noor et al. (2010) there are many iterative methods based on second degree equation

Barret et al. (1994) stated that the term "iterative method" refers to a wide range of techniques that use successive approximations to obtain more accurate solutions to a linear system at each step.

Numerical iterative methods developed by different researchers to solve problems of nonlinear systems we encounter in everyday life.

Darvishi and Barati(2007) derived a supper cubic iterative method from the Andomian decomposition to solve nonlinear equations. But Babejee et al.(2008)introduced a note on the local convergence of iterative methods based on the Andomian decomposition and 3-node quadrature rule. While Chun (2006) presented a new iterative method to solve nonlinear equations by improving Newton's method. Using Andomian decomposition, Darvishi and Barati(2007)and Darvishi and Barati(2007) has constructed new methods. Hafiz and Bahgat(2012) created a new method using modified Householder iterative method for solving system of non-linear equations. Golbabai and Javidi(2007) have applied the homotopy perturbation method to build a new family of Newton like iterative methods for solving system of nonlinear equations. Hosseini(2010)developed an improved homotopy analysis method for solving a system of nonlinear algebraic equations and compared this method with homotopy perturbation method and Newton-Raphson method, this method converges faster than both methods compared with it along with being effective performance and convenient.

Broyden(1965) formulated the class of iterative methods for solving systems of nonlinear equations, because of challenges he faced while using Newton method. In this work we limit ourselves to only three of these methods which are Muller method, Chebyshev Method and the Multi-Point Iteration method.

Methods such as the Muller method, Chebyshev Method and Multi-Point Iteration method are called roots finding algorithms which work on continuous functions. These methods are always convergent since they are based on reducing the interval between the three guesses so as to zero in on the root of the equation.

Statement of the Problem

In ages past algebraic methods such as the substitution method, the elimination method, quadratic formula and factorization are used in solving equations. In Linear Algebra, we learned that solving systems of linear equations can be implemented by using row reduction as an algorithm. However, when these methods are not successful, Numerical methods are used to approximate solutions of equations when exact solutions cannot be determined via algebraic methods. Several numerical methods could be used to achieve this. The question is which of the numerous methods is most effective and accurate.

Aim

The aim of this project is to study some formulas used for the solution of some selected iterative methods base on second degree equation.

Objectives

1. Derive the formulas for the selected iterative methods.
2. Use the derived formulas to solve nonlinear equations.
3. Identify which of the method is faster in getting convergence to the exact root.

2. METHODS

2.1. Derivation of Muller's Method

The Muller's method is based on the principle that if $x_{i-2},\ x_{i-1}\ and\ x_i$ are three approximations to the root of $f(x) = 0$, then we may determine $a_0,\ a_1$ and a_2 in equation (1.1) by using the conditions in equation (1.1)

$$\left.\begin{aligned} f(x_{i-2}) &= a_0 x_{i-2}^{\ 2} + a_1 x_{i-2} + a_2 \\ f(x_{i-1}) &= a_0 x_{i-1}^{\ 2} + a_1 x_{i-1} + a_2 \\ f(x_i) &= a_0 x_i^{\ 2} + a_1 x_i + a_2 \\ f(x) &= a_0 x^2 + a_1 x + a_2 \end{aligned}\right\} \tag{2.1.1}$$

Eliminating $a_0,\ a_1$ and a_2 in equation (2.1.1) we get

$$\begin{vmatrix} f(x_{i-2}) = x_{i-2}^2 x_{i-2}\, 1 \\ f(x_{i-1}) = x_{i-1}^2\ x_{i-1}\, 1 \\ f(x_i) = x_i^2 x_i\, 1 \\ f(x) = x^2 x 1 \end{vmatrix} = 0 \tag{2.1.2}$$

Which we may simplify to obtain f(x)

$$f(x) = \frac{(x-x_{i-1})(x-x_i)}{(x_{i-2}-x_{i-1})(x_{i-2}-x_i)} + f_{i-2}\frac{(x-x_{i-2})(x-x_i)}{(x_{i-1}-x_{i-2})(x_{i-1}-x_i)} + f_{i-1}\frac{(x-x_{i-2})(x-x_{i-1})}{(x_i-x_{i-2})(x_i-x_{i-1})}f_i = 0 \quad (2.1.3)$$

Equation (2.1.3) may also be written as

$$\frac{h(h+h_i)}{h_{i-1}(h_{i-1}+h_i)} + f_{i-2}\frac{h(h+h_i+h_{i-1})}{(h_i h_{i-1})} + f_{i-1}\frac{(h+h_i)(h+h_i+h_{i-1})}{h_i(h_i+h_{i-1})}f_i = 0 \quad (2.1.4)$$

Where

$$h = x - x_i \text{and} h_i = x_i - x_{-1}$$

We further obtain

$\lambda = \frac{h}{h_i}$, $\lambda_i = \frac{h_i}{h_{i-1}}$ and $\delta_i = 1 + \lambda_i$

Express equation (2.1.4) in the form we obtain

$$\lambda^2 C_i + \lambda g_i + \delta_i f_i = 0 \quad (2.1.5)$$

Where our:

$$g_i = \lambda^2 f_{i-2} - \delta_i^2 f_{i-1} + (\lambda_i + \delta_i)f_i$$

$$C_i = \lambda_i(\lambda_i f_{i-2} - \delta_i f_{i-1} + f_i)$$

Solving equation (2.1.5) for λ, we obtain

$$\lambda = -g_i \pm \frac{\sqrt{g_i^2 - 4\delta_i f_i C_i}}{2C_i} \text{ or } \lambda_{i+1} = \frac{-2\delta_i f_i}{g_i \pm \sqrt{g_i^2 - 4\delta_i f_i C_i}} \quad (2.1.6)$$

The sign in the denominator in equation (2.1.6) is chosen such that λ_{i+1} has the smallest absolute value. Thus, we have $\lambda_{i+1} = \frac{h}{h_i} = \frac{x-x_i}{x_i-x_{i-1}}$

$$x = x_i + (x_i + x_{i-1})\lambda_{i+1} \quad (2.1.7)$$

Replacing x on the left-hand side of equation (3.1.7) by x_{i+1}

$$x_{i+1} = x_i + (x_i + x_{i-2})\lambda_{i+1} \quad (2.1.8)$$

Equation (2.1.8) is called the Muller's formula for solving second degree nonlinear equations of the form $f(x) = 0$

So, starting with an initial guessx_i, one can find the next guess x_{i+1} by using equation (2.1.3). You can repeat this process until you can find a desirable tolerance.

2.2. Derivation of Chebyshev Method

We determine a_0, a_1 and a_2 in equation (1.1.1)

$$\left.\begin{aligned} f_i &= a_0 x_i^2 + a_1 x_i + a_2 \\ f_i' &= 2a_0 x_i + a_1 \\ f_i'' &= 2a_0 \end{aligned}\right\} \quad (2.2.1)$$

On eliminating $a_1'$from equation (1.1.1) and equation (2.2.1), we obtain

$$f_i + (x - x_i)f_i' + (x - x_i)f_i'' = 0 \quad (2.2.2)$$

Which is the Taylor's series expansion of f(x) about$x = x_i$, such that the terms of order$(x - x_i)^3$ and higher power. The equation (2.2.2) is a quadratic equation and can be solved easily.

In order to get the correct root approximation, we write equation (2.2.2) as

$$x_{i+1} = \frac{-f_i}{f_i'} - \frac{1}{2}(x_{i+1} - x_i)^2 \frac{f_i''}{f_i'} \quad (2.2.3)$$

We substitute $x_{i+1} - x_i$ from$x_{k\frac{-f_k}{f_k'}}$, $k = 0,1$ ...by $\frac{-f_k}{f_k'}$on the right-hand side and obtain

$$x_{i+1} = \frac{-f_i}{f_i'} - \frac{1}{2}\left(\frac{-f_i}{f_i'}\right)^2 \frac{f_i''}{f_i'} \quad (2.2.4)$$

$$x_{i+1} = \frac{-f_i}{f_i'} - \frac{1}{2}\frac{-f_i^{\,2}}{(f_i')^3}\cdot\frac{f_i''}{f_i'} \quad (2.2.5)$$

Equation (2. 2. 5) is called the Chebyshev method for solving equation of the form $f(x) = 0$

This method requires three evaluations for every iteration.

2.3. Derivation of Multi-Point Iteration Method

We write the equation (3.2.2) as $x_{i+1} - x_i = \frac{-f_i}{f_i' + \frac{1}{2}(x_{i+1} - x_i)f_i''}$

$$x_{i+1} - x_i \cong \frac{-f_i}{f_i'(x_i + \frac{1}{2}(x_i - x_i)} \quad (2.3.1)$$

And again replace $x_i - x_i$ by $\frac{-f_k}{f_k'}$ in the right-hand side of (2.3.1)

We have: $x_{i+1} - x_i \cong \frac{-f_i}{f_i'\left(x_i + \frac{1}{2}\frac{f_k}{f_k'}\right)} k = 0,1,2,$ (2.3.2)

Equation (2. 3. 2) is called the Multi-Point method for solving equation of the form $f(x) = 0$

For computational purpose we may compute equation (2.3.2) as the two-stage method.

$$\left.\begin{aligned} x_{i+1}^* &= x_i - \frac{f_k}{f_k'} \\ x_{i+1} &= x_{k+1}^* - \frac{f_{i+1}^*}{f_i'} \end{aligned}\right\} \quad (2.3.3)$$

3. RESULTS AND DISCUSSION

3.1. Convergence Analysis

An iterative method is said to be of order P or has the rate of convergence p, if p is the largest positive real number for which there exist a finite constant $c \neq 0$ such that,

$|E_{k+1}| \leq c \,|\propto_k|\, p$

Where $E_k = E_k - \propto$ *is* the error in the x^{th} iterate. The constant c is called the error constant and usually depends on derivative of $f(x)$ at $x = \propto$.

3.2. Root

The root of a nonlinear equation f(x) = 0 using iterative method is the value of the iteration number which the digit stops changing to the required degree of accuracy.

3.3. Numerical Examples

Problem 1

Use the three methods derived to find the root of the equation $f(x) = x^3 - 5x + 1 = 0$ correct to Six decimal places.

Using Microsoft-Excel 2007 to Solve

3.3.1. Muller's Method

Steps:

1. Enter n in cellA_1, Enter$x$ in cellB_1, Enter$f(x)$ in cell C_1 , Enter h in cell D_1, Enter lambda in cellE_1, Enter d in cellF_1, Enter g in cellG_1, Enter C in cellH_1, Enter Lambda in cellI_1, Enter x + 1 in cellJ_1
2. Enter the Value 0 to 4 in column A_2 to A_{10} using the edit – fill- series sequence of command. These are the iteration numbers.
3. Enter the initial value of$x$ in cell B_2, in cell B_3 , in cell B_4 and in cell B_5 enter the formula

   = J4 and copy content of B_5 into B_6 toB_{10}.
4. Enter the formula $= (POWER(B2,3) - 5 \times B2 + 1)$in cell$C_2$. Now copy the content of cell C_2 into C_3 to C_{10}.
5. Enter the formula $= (B3 - B2)$ in cell D_2 and copy the content of cellD_2 into D_3 toD_{10}.

6. Enter the formula $= (D4/D3)$ in cell E_4 and copy the content of cellE_4 into E_5 toE_{10}.
7. Enter the formula $= (1 + E4)$in cell F_4 and copy the content of cellF_4 into F_5 toF_{10}.
8. Enter the formula =(POWER(E4,2) ×C2-POWER(F4,2) ×C3+(E4+F4) ×C4) in cell G_4 and copy the content of cellG_4 into G_5 toG_{10}.
9. Enter the formula $= (C2 \times ((E4 \times C2) - (F4 \times C3) + C4))$ in cell H_4 and copy the content of cellH_4 into H_5 toH_{10}.
10. Enter the formula $= ((-2 \times F4 \times C4)/((G4) - SQRT(POWER(G4{,}2) - (4 \times F4 \times H4 \times C4))))$ in cell I_5 and copy the content of cellI_5 into I_6 toI_{10}.
11. Enter the formula $= (B4 + (B4 - B3) \times I5)$in cell J_4 and copy the content of cellJ_4 into J_5 toJ_{10}.

3.3.2. Chebyshev Method

Steps:

12. Entern in cell A_1, enter x in cell B_1, enter f(x) in cell C_1, enter f' (x) in cell D_1, enter f'' (x) in cell E_1,and enter $\mathrm{x} + 1$ in cell F_1.
13. Enter the value 0 to 8 in column Afrom A_2 to A_{10} using the edit-fill-series sequence of command. Enter the initial values x in cell B_2 as 0.5 and in cell B_3asF_2.
14. Enter the formula $= (\mathrm{POWER(B2{,}3)} - 5 \times (\mathrm{B2}) + 1)$ in cell C_2 now copy the content of C_2 into C_3 toC_{10}.
15. Enter the formula $= (3 \times \mathrm{POWER(B2{,}2)} - 5)$into cell$\mathrm{D}_4$. Now copy the content of cell D_4into D_5 toD_{10}.
16. Enter the formula $= (6 \times \mathrm{B3})$into cellE_4. Now copy the content of cell E_4into E_5 toE_{10}.
17. Enter the formula =(B2-(C2/D2)-0.5×(POWER(C2,2)/POWER(D2,3)) ×E2)
18. Into cellF_4. Now copy the content of cell F_4into F_5 toF_{10}.

3.3.3. Multi-Point Method

Steps:

19. Entern in cell A_1, enter x in cell B_1, enter f(x) inf cell C_1, enter f' (x) in cell D_1, enter $\mathrm{x} + 1$ in cell E_1, and enter f'^{*}(x)in cell F_1.
20. Enter the value 0 to 8 in column A_2 toA_{10} using the edit –fill-series sequence of command. These are the iteration numbers.
21. Enter the initial values x in cell B_2 and in cell B_3 as $= (\mathrm{E2} - (\mathrm{F2}/\mathrm{D2}))$. And in cells B_4 =E3 using the edit-fill series sequence to extend to B_{10}
22. Enter the formula $= (\mathrm{POWER(B2{,}3)} - 5 \times (\mathrm{B2}) + 1)$ in cell C_2 . now copy the content of C_2 into cell C_3 to C_{10}.
23. Enter the formula $= (3 \times \mathrm{POWER(B2{,}2)} - 5)$into cell D_4. Now copy the content of cell D_4 into D_5 to D_{10}.
24. Enter the formula $= (\mathrm{B2} - (\mathrm{C2}/\mathrm{D2}))$ into cellE_4. Now copy the content of cell E_4 into E_5 to E_{10}.
25. Enter the formula $= (\mathrm{POWER(E2{,}3)} - 5 \times (\mathrm{E2}) + 1)$into cell$\mathrm{F}_4$. Now copy the content of cell F_4 into F_5 to F_{10}.

Solution:

Numbers of iterations (n)	Muller's Method	Chebyshev Method	Multi-Point Method
0	0.000000	0.213413	0.176471
1	0.500000	0.201640	0.201638
2	1.000000	0.201640	0.201640
3	0.191857	0.201640	0.201640
4	0.201179	0.201640	0.201640
5	0.201657	0.201640	0.201640
6	0.201640	0.201640	0.201640
7	0.201640	0.201640	0.201640
8	0.201640	0.201640	0.201640

Problem 2

Use the three methods derived to finds the root of the equation $f(x) = x^3 - x - 4 = 0$, correct to six decimal places.

Solution:

Numbers of iterations (n)	Muller Method	Chebyshev Method	Multi-point Method
0	1.000000	1.762678	1.500000
1	1.500000	1.796295	1.753900
2	2.000000	1.796322	1.797482
3	1.848744	1.796322	1.796323
4	1.778084	1.796322	1.796322
5	1.795685	1.796322	1.796322
6	1.796323	1.796322	1.796322
7	1.796322	1.796322	1.796322
8	1.796322	1.796322	1.796322

Problem 3

Use the three methods derived to find the root of the equation$f(x) = x^4 - x - 10 = 0$, correct to six decimal places.

Solution:

Numbers of iterations (n)	Muller Method	Chebyshev Method	Multi-point Method
0	0.000000	2.000000	2.000000
1	1.000000	1.864523	1.929209
2	2.000000	1.855616	1.859841
3	1.795597	1.855585	1.855600
4	1.853618	1.855585	1.855585
5	1.855579	1.855585	1.855585
6	1.855585	1.855585	1.855585
7	1.855585	1.855585	1.855585
8	1.855585	1.855585	1.855585

4. Discussion of Result

We have considered some numerical problems in this project. The problems were solved using the derived methods in chapter three of this project.

Based on the result from problem 1, 2 and 3 in this project, it was found out that Chebyshev method converges to the root $\propto$ faster than the Multi-point method while Muller method is the slowest among all. The results from the three numerical problems in this project have proved that Chebyshev method converges faster than the remaining two methods.

5. Conclusion

This work has compared the rate of convergence of three iterative methods for solving second degree nonlinear equations

We discovered that the fastest method in getting convergence to the exact root is the Chebyshev Method followed closely by the multi-point method while the Muller method is the slowest of all three methods studied. The Muller method was hardest method to implement with twelve steps needed while the multi-point method was easier to implement having only seven steps with Chebyshev method being the easiest with only six steps needed.

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