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Blow-Up for Some Wave Equations with a Derivative Nonlinearity

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Abstract: *Sufficient conditions for blow-up of solutions to the initial-boundary value problem for some wave equations with a derivative nonlinearity are established by eigenfunction method. This extend the early results.*

Keywords: *wave equation; blow-up; initial-boundary value problem; derivative nonlinearity*

1. Introduction

In this note, we are concerned with the initial boundary value problem of the following types:

$$u_{tt}-\Delta u=\mu|\nabla u|^{p},\ \mu>0,(x,t)\in\Omega\times(0,T), \tag{1.1}$$

$$u(x,t)=0,\ \ x\in\partial\Omega,t\in(0,T), \tag{1.2}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),\ \ x\in\Omega; \tag{1.3}$$

$$u_{tt}-\Delta u+|u|^{q}=\mu|\nabla u|^{p},\ \ \mu>0,(x,t)\in\Omega\times(0,T), \tag{1.4}$$

$$u(x,t)=0,\ \ x\in\partial\Omega,t\in(0,T), \tag{1.5}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),\ \ x\in\Omega \tag{1.6}$$

and

$$u_{tt}-\Delta u+u_t=\mu|\nabla u|^{p},\ \ \mu>0,(x,t)\in\Omega\times(0,T), \tag{1.7}$$

$$u(x,t)=0,\ \ x\in\partial\Omega,t\in(0,T), \tag{1.8}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),\ \ x\in\Omega, \tag{1.9}$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary $\partial\Omega$ and $p,q>0$. Our main goal is to find sufficient conditions forblow up of solutions to problem (1.1)-(1.3), problem (1.4}-(1.6) and problem (1.7)-(1.9).

The following nonlinear wave equation

$$u_{tt}-\Delta u = f(u,u_t,\nabla u) \tag{1.10}$$

attracted attention of the researchers for many years. The case when

$f(u,u_t,\nabla u) = -a|u_t|^m u_t + b|u|^p u$ (or $f(u) = b|u|^p$, here $a \geq 0, b \geq 0$), e.g.

$$u_{tt}-\Delta u + a|u_t|^m u_t = b|u|^p u \tag{1.11}$$

has been extensively studied over the past decades. It is well known that the nonlinear term $|u|^p u$ drives the solution of (1.11) to blow up in finite time. Various sufficient conditions for blowup have been provided and qualitative properties have also been investigated (see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10], to cite just a few). By contrast, there has been a relatively small number of studies of blowup for nonlinearities with a dependence on spatial derivatives of u. Ebihara [11, 12, 13] established global existence of classical solutions and asymptotic behavior of solutions of equation (1.10). When $f(u,u_t,\nabla u) = -a(x)\beta(u_t,\nabla u)$ in (1.10), Slemrod [14], Vancostenoble [15] and Haraux [16] proved the weak asymptotic stabilization of solutions. Quite recently, Nakao [17, 18, 19, 20, 21] considered the nonlinear wave equations of the form

$$u_{tt}-\Delta u + \rho(x,u_t) = f(u,u_t,\nabla u), \tag{1.12}$$

and he proved the global existence and decay of solutions. There seems to be little investigations concerning the blow-up of solutions for equation (1.10) when the nonlinear perturbation term f depends on the derivatives of u. This is because in the case it seems difficult to handle the term $|\nabla u|^p$. Sideris [22] has shown blow-up of small data solutions in finite time for the Cauchy problem of equation (1.10) with $f(u,u_t,\nabla u) = a^2|\nabla u|^2 + b^2|\Delta u|^2$ in three dimensions. As far as we are aware, this is the first blow-up result for equation (1.10) when the nonlinear perturbation term f depends on only the derivatives of u. Then the result was extended by Schaeffer [23]and Rammaha [24, 25].However, very little is known in the literature concerning the blow-up of solutions for initial boundary problem of equation (1.10) when the nonlinear perturbation term f depends on only the derivatives of u and such a method in [22, 23, 24, 25] cannot be used in this case.

The object of this paper is to show the sufficient conditions for blow-up of solutions for the initial boundary value problem of equation (1.10) with $f(u,u_t,\nabla u) = \mu|\nabla u|^p$ or

$f(u,u_t,\nabla u)=\mu|\nabla u|^p-|u|^q$. The eigen function method developed in [1] was used here. We also extend Lemma 1.1 in [1] and then get the sufficient conditions for blow-up of solutions to the initial boundary value problem (1.7)-(1.9). This method applies also to the case of the equation (1.10) with Neumann boundary condition.

2. Main Results

Throughout this paper we assume all function spaces are considered over real field and their notations and definitions are same as [26]. For simplicity, we take $\mu=1$. By the usual Galerk in method and similar to the proof in [11], we can obtain regular solution in the local sense. Now we mention some Lemmas which play an essential role in this paper.

Lemma 1 [1] Let $\phi(t)\in C^2$ satisfy

$$\phi_{tt}\ge h(\phi),\quad t\ge 0$$

with $\phi(0)=\alpha>0,\phi_t(0)=\beta>0$. Suppose that $h(s)\ge 0$ for all $s\ge\alpha$. Then, $\phi_t(t)>0$ where $\phi_t(t)$ exists and the following inequality holds

$$t\le\int^{\phi(t)}\alpha[\beta^2+2\int_\alpha^s h(v)dv]^{-1/2}ds.$$

We consider the following spectral problem

$$\Delta w+\lambda w=0\quad in\quad\Omega, \tag{2.1}$$

$$w=0,\quad on\quad\partial\Omega., \tag{2.2}$$

It is well known that problem (2.1)-(2.2) has the smallest eigen value $\lambda_1>0$ with the corresponding normalized eigen function $w_1>0$ in Ω, $\int_\Omega w_1(x)dx=1$. Then we denote

$$k_0=\left(\int_\Omega\frac{|\nabla w_1|^{\frac{p}{p-1}}}{w_1^{1/(p-1)}}dx\right)^{\frac{p-1}{p}}. \tag{2.3}$$

Theorem 2 Suppose $p>1$. Let $u(x,t)$ be a regular solution of problem (1.1)-(1.3). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x)w_1(x)dx=\alpha,\int_\Omega u_1(x)\psi_1(x)dx=\beta,$$

where $\alpha>\dfrac{k_0^{p/(p-1)}}{\lambda_1}>0,\beta>0$. Then, the solution of problem (1.1)-(1.3) blows up in a finite time.

Proof Let

$$U(t)=\int_\Omega u(x,t)w_1(x)dx.$$

Then $U(0)=\alpha>0,U_t(0)=\beta>0$ and as it follows from (1.1)-(1.3), $U(t)$ satisfies

$$U_{tt}+\lambda_1 U=\int_\Omega|\nabla u|^p\ w_1dx. \tag{2.4}$$

By (2.1) and Holder inequality, we get

$$\lambda_1 U\le\left|\int_\Omega u(x,t)\lambda_1 w_1(x)dx\right|=\left|\int_\Omega u(x,t)\Delta w_1(x)dx\right|$$

$$=|\int_\Omega \nabla u \nabla w_1 dx| \le \int_\Omega |\nabla u||\nabla w_1| dx = \int_\Omega (|\nabla u| w_1^{1/p}) \frac{|\nabla w_1|}{w_1^{1/p}} dx$$

$$\le (\int_\Omega \frac{|\nabla w_1|^{\frac{p}{p-1}}}{w_1^{1/(p-1)}} dx)^{\frac{p-1}{p}} (\int_\Omega |\nabla u|^p w_1 dx)^{\frac{1}{p}} = k_0 (\int_\Omega |\nabla u|^p w_1 dx)^{\frac{1}{p}},$$

that is to say

$$\int_\Omega |\nabla u|^p w_1 dx \ge (\frac{\lambda_1}{k_0})^p U^p. \tag{2.5}$$

Therefore, from (2.4) and inequality (2.5), we obtain the ordinary differential inequality

$$U_{tt} \ge (\frac{\lambda_1}{k_0})^p U^p - \lambda_1 U, \tag{2.6}$$

with $U(0)=\alpha>0, U_t(0)=\beta>0$. Denote $h(s)=(\frac{\lambda_1}{k_0})^p s^p - \lambda_1 s$,since $h(s)>0$ for $s \ge \alpha$, it follows from Lemma 1 that

$$t \le \int_\alpha^{U(t)} [\beta^2 - \lambda_1 s^2 + \frac{2}{p+2} (\frac{\lambda_1}{k_0})^p (s^{p+1} - \alpha^{p+1})]^{-1/2} ds,$$

and $U(t)$ develops a singularity in finite time $t_0 \le \overline{T}$, where

$$\overline{T} = \int_\alpha^\infty [\beta^2 - \lambda_1 s^2 + \frac{2}{p+2} (\frac{\lambda_1}{k_0})^p (s^{p+1} - \alpha^{p+1})]^{-1/2} ds.$$

Finally, since $U(t)>0$, we have $U(t)=|U(t)| \le sup_\Omega |u(x,t)| \int_\Omega w_1 dx \le sup_\Omega |u(x,t)|$,which proves the theorem.

Theorem 3 Suppose $p \ge 2, 0<q<2$. Let $u(x,t)$ be a regular solution of problem (1.4)-(1.6). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x) w_1(x) dx = \alpha_1, \int_\Omega u_1(x) \psi_1(x) dx = \beta_1,$$

where α_1 is the positive root of the equation $h_1(s)=(\frac{\lambda_1}{k_0})^p s^p - s^q - \lambda_1 s = 0$ and $\beta_1>0$. Then, the solution of problem (1.4)-(1.6) blows up in a finite time.

Proof Let

$$U(t) = \int_\Omega u(x,t) w_1(x) dx.$$

Then $U(0)=\alpha_1>0, U_t(0)=\beta_1>0$ and as it follows from (1.4)-(1.6), $u(x,t)$ satisfies

$$U_{tt} + \lambda_1 U + \int_\Omega |u|^q w_1 dx = \int_\Omega |\nabla u|^p w_1 dx. \tag{2.7}$$

Then (2.5) and the inequality $\int_\Omega |u|^q w_1 dx \ge U^q$ yield the ordinary differential inequality

$$U_{tt} \ge (\frac{\lambda_1}{k_0})^p U^p - U^q - \lambda_1 U = h_1(U), \tag{2.8}$$

with $U(0)=\alpha_1>0, U_t(0)=\beta_1>0$. Since $h_1(s)>0$ for $s \ge \alpha_1$. The rest of the proof is similar to the proof of Theorem 2 and the proof is complete.

Now we extend Lemma 1 (see Lemma 1.1 in [1] and [3]) to the following Theorem .

Lemma 4 Let $\phi(t) \in C^2$ satisfy

$$\phi_{tt} + k_1 \phi_t \ge h(\phi), \quad t \ge 0, \tag{2.9}$$

with $\phi(0)=\alpha>0, \phi_t(0)=\beta>0$, where $k_1>0$. Suppose that $h(s)\geq 0$ for all $s\geq\alpha$. If

$\delta_0=k_1\int_\alpha^{+\infty}[\beta^2+2\int_\alpha^s h(\rho)d\rho]^{-\frac{1}{2}}ds<1$,

then $\phi_t(t)>0$ where $\phi_t(t)$ exists and $\lim_{t\to T^-}\phi(t)=+\infty$ where $T\leq T^*=-\frac{1}{k_1}ln(1-\delta_0)$.

Proof Because $\phi(0)=\alpha>0, \phi_t(0)=\beta>0$, then there exist an interval $[0,T_0)$ such that $\phi_t(t)>0$ and $\phi(t)=\alpha>0$ for $t\in[0,T_0)$. If it is false, let

$$t_1=\inf\{t:\phi(t)=\alpha\}, t_2=\inf\{t:\phi_t(t)=0\}.$$

If $t_2<t_1$, taking into account the condition (2.9) and the fact that $h(s)\geq 0$ for all $s\geq\alpha$, we have

$$\frac{d}{dt}(e^{k_1 t}\phi_t)=e^{k_1 t}(\phi_{tt}+k_1\phi_t)\geq e^{k_1 t}h(\phi)>0.$$

Thus $\phi_t(t_2)>e^{-k_1 t_2}\phi_t(0)>0$, which contradicts $\phi_t(t_2)=0$, and so we have $t_2\geq t_1$. Furthermore, we have $\phi_t(t)>0$ for $t\in[0,t_1)$.In this case, we get that $\phi(t_1)=\phi(0)+\int_0^{t_1}\phi_t(s)ds>\phi(0)=\alpha>0$, this is a contradiction of the fact $\phi(t_1)=\alpha$. Thus, there exist an interval $[0,T_0)$ such that $\phi_t(t)>0$ and $\phi(t)>\alpha$ for $[0,T_0)$.

A multiplication of (2.9) by $2e^{2k_1 t}\phi_t(t)$ gives

$$2e^{2k_1 t}\phi_t\phi_{tt}+2k_1e^{2k_1 t}(\phi_t)^2\geq 2e^{2k_1 t}h(\phi)\phi_t,$$

that is,

$$\frac{d}{dt}[e^{2k_1 t}(\phi_t)^2]\geq 2e^{2k_1 t}h(\phi)\phi_t\geq 2h(\phi)\phi_t=2\frac{d}{dt}\int_\alpha^\phi h(s)ds. \qquad (2.10)$$

Integrating (2.10) from 0 to t yields

$$e^{2k_1 t}(\phi_t)^2-(\phi_t(0))^2\geq 2\int_\alpha^\phi h(s)ds,$$

since $\phi_t>0$, hence

$$\phi_t\geq e^{-k_1 t}(\beta^2+2\int_\alpha^\phi h(s)ds)^{-\frac{1}{2}}. \qquad (2.11)$$

For (2.11), we may separate variables and integrate over $(0,t)$ to obtain

$$1-e^{-k_1 t}\leq k_1\int_\alpha^{+\infty}(\beta^2+2\int_\alpha^y h(s)ds)^{-\frac{1}{2}}dy=\delta_0.$$

Therefore, we get the result.

Theorem 5 Suppose $p>1$. Let $u(x,t)$ be a regular solution of problem (1.7)-(1.9). Suppose that the following conditions are satisfied:

$$\int_\Omega u_0(x)w_1(x)dx=\alpha_2, \int_\Omega u_1(x)\psi_1(x)dx=\beta_2,$$

where α_2 is the positive root of the equation $h_2(s)=(\frac{\lambda_1}{k_0})^p s^p-\lambda_1 s=0$ and $\beta_2>0$. If $\delta_1=k_1\int_{\alpha_2}^{+\infty}[\beta_2^2+2\int_{\alpha_2}^s h_2(\rho)d\rho]^{-\frac{1}{2}}ds<1$, then the solution of problem (1.7)-(1.9) blows up in a finite time.

Proof Similar to the proof of Theorem 2, we can obtain the ordinary differential inequality

$$U_{tt}+U_t+\lambda_1 U-(\frac{\lambda_1}{k_0})^p U^p\geq 0, \qquad (2.12)$$

with $U(0)=\alpha_2>0, U_t(0)=\beta_2>0$.Denote $h_2(s)=(\frac{\lambda_1}{k_0})^p s^p-\lambda_1 s$, since $h_2(s)>0$ for $s\geq\alpha_2$, it follows from Theorem $\lim_{t\to T_0^-} U(t)=\infty$, for some $T_0 \le T^* = -\frac{1}{c} ln(1-\delta_1)$. Furthermore, since $U(t)>0$, we have

$U(t) = |U(t)| \le sup_\Omega |u(x,t)| \int_\Omega w_1 dx \le sup_\Omega |u(x,t)|$,

and we get

$\lim_{t\to T_0^-} \| u \|_p^p = \infty, \forall 1 \le p \le \infty$,

for some $T_0 \le T^* = -\frac{1}{c} ln(1-\delta_1)$, which proves the theorem.

Remark 1 The same results hold for the equation (1.10) with the boundary condition $a\frac{\partial u}{\partial n} + bu = 0$.

Remark 2 It seems that the method can also be applied to the equation (1.10) when Δu is replaced by $p-Laplace$ operator $div(|\nabla u|^p \nabla u)$ and it seems that the method can be applied to equation (1.10) with $f(u,u_t,\nabla u) = |\nabla u|^p + \Delta u_t$.

References

[1] R. T. Glassey. Blow-up theorems for nonlinear wave equations. Mathematische Zeitschrift, 1973, 132(3): 183-203.

[2] H.A.Levine. Instability and nonexistence of global solutions to nonlinear wave equations of theform $Pu_{tt} = Au + F(u)$. Transactions of the American Mathematical Society, 1974, 192: 1-21.

[3] H.A. Levine. Nonexistence of global solutions to some properly and improperly posed problem of mathematical physics: The method of unbounded Fourier coefficients. Mathematische Annalen, 1975, 214(3): 205-220.

[4] L.E. Payne, D. Sattinger. Saddle points and instability of nonlinear hyperbolic equations. Israel Journal of Mathematics, 1975, 22(3-4): 273-303.

[5] V.K. Kalantarov, O.A. Ladyzhenskaya. The occurrence of collapse for quasilinear equations of parabolic andhyperbolic types. Journal of Soviet Mathematics, 1978, 10(1): 77-102.

[6] V. Georgiev, G. Todorova. Existence of a solution of the wave equation with nonlinear damping and source terms.Journal of Differential Equations, 1994, 109: 295-308.

[7] S. A. Messaoudi. Blow-up in a nonlinear damped wave equation. Mathematische Nachrichten, 2001, 231(1): 105-111.

[8] V. Barbu, I.Lasiecka, M.A. Rammaha. On nonlinear wave equations with degenerate damping and source terms. Transactions of the American Mathematical Society, 2005, 357(7): 2571-2611.

[9] E.Vitillaro. Global nonexistence theorems for a class of evolution equationwith dissipation. Archive for Rational Mechanics and Analysis, 1999, 149(2): 155-182.

[10] M.O. Korpusov, A.A. Panin. Blow-up of solutions of an abstract Cauchy problem

[11] for a formally hyperbolic equation with double non-linearity. Izvestiya: Mathematics, 2014, 78(5): 937-985.

[12] Y. Ebihara. On the local classical solution of the mixed problem for nonlinear wave equation. Mathematical reports of College of General Education, Kyushu University, 1974, 9(2): 43-65.

[13] Y. Ebihara. On the equations with variable coefficients $u_{tt} - u_{xx} = F(x,t,u,u_x,u_t)$. Funkcialaj Ekvacioj, 1977, 20(1): 77-95.

[14] Y. Ebihara. Nonlinear wave equations with variable coefficients. Funkcialaj Ekvacioj, 1979, 22(2): 143-159.

[15] M. Slemrod. Weak asymptotic decay via a "relaxed invariance principle" for a wave equation with nonlinear, nonmonotonedamping. Proceedings of the Royal Society of Edinburgh Sect. A, 1989, 113(1): 87-97.

[16] J. Vancostenoble. Weak asymptotic decay for a wave equation with gradient dependent damping. Asymptotic Analysis, 2001, 26: 1-20.

A. Haraux. Remarks on weak stabilization of semilinear wave equations. ESAIM: Control, Optimization and Calculus of Variations, 2001,6: 553-560.

[17] M. Nakao. Existence of global decaying solution to the Cauchy problem for a nonlinear dissipative wave equations of Klein-Gordon type with aderivative nonlinearity. Funkcialaj Ekvacioj, 2012, 55: 457-477.

[18] M. Nakao. Global existence and decay for nonlinear dissipative wave equations with a derivative nonlinearity. Nonlinear Analysis, 2012, 75: 2236-2248.

[19] M.Nakao. Existence of global decaying solutions to the exterior problem for the Klein-Gordon equation with a nonlinear localized dissipation and a derivative nonlinearity. Journal of Differential Equations, 2013, 255: 3940-3970.

[20] M. Nakao. Global solutions to the initial-boundary value problem for the quasilinear viscoelastic equation with a derivative nonlinearity. Opuscula Mathatics, 2014, 34(3): 569-590.

[21] M. Nakao. Existence of global decaying solutions to the initial boundary value problem for the quasi linear wave equation of p-Laplacian type with Kelvin-Voigt dissipation and a derivative nonlinearity. Kyushu Journal of Mathematics, 2016,70: 63-82.

[22] T. Sideris. Global behavior of solutions to nonlinear wave equations in three dimensions. Communications in Partial Differential Equations, 1983, 8(12):1291-1323.

[23] J. Schaeffer. Finite-time blow-up for $u_{tt} - \Delta u = H(u_r, u_t)$ \$ in two space dimensions. Communications in Partial Differential Equations, 1986, 11(5):513-543.

[24] M.A. Rammaha. Finite-time blow-up for nonlinear wave equations in high dimensions. Communications in Partial Differential Equations, 1987, 12(6):677-700.

[25] M.A. Rammaha. On the blowing up solutions to nonlinear wave equations in two space dimensions. Journal fur Die Reine Und Angewandte Mathematik, 1988, 391: 55-64.

[26] J.L. Lions. Quelquesmethodes de resolution des problems aurlimites non lineaires. Dunod Gauthier-villars, Paris, 1969.

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