International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 6, Issue 5, 2018, PP 46-51
ISSN No. (Print) 2347-307X & ISSN No. (Online) 2347-3142
DOI: http://dx.doi.org/10.20431/2347-3142.0605005
www.arcjournals.org

More on Square and Square Root of a Node on T3 Tree

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Abstract: *The article proves several inequalities derived from nodal multiplication on T3 tree. The proved inequalities are helpful to estimate certain quantities related with the T3 tree as well as examples of proving an inequality embedded with the floor functions.*

Keywords: *Square, Square root, Integer, Binary tree*

1. INTRODUCTION

The square and the square root are undoubtedly very important operations for a number. As a new structure of odd integers, the T3 tree, which was introduced in [1] and [2], is of course necessary to make clear these two operations. In fact, for a given node in the tree, the problem where its square and its square root locate is surely a fundamental problem. Paper [2] mentioned some general properties of the square of a node in $\boldsymbol{T}_3$, paper [4] was the first one to disclose the properties of the square root of a node. However, one can see that, there are still a lot of unknown properties. This paper shows a little more of the properties related with the square and the square root of a node.

2. PRELIMINARIES

This section lists for later sections the necessary preliminaries, which include definitions, notations and lemmas.

2.1. Definitions and Notations

Let S be a set of finite positive integers with s_0 and s_n being the smallest and the biggest nodes respectively; an integer x is said to be clamped in S if $s_0 \le x \le s_n$. Symbol $x \,\square\, S$ indicates that x is clamped in S. Symbol $\lfloor x \rfloor$ is the floor function, an integer function of real number x that satisfies inequality $x-1<\lfloor x \rfloor \le x$, or equivalently $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.

In this whole paper, symbol $\boldsymbol{T}_3$ is the $\boldsymbol{T}_3$ tree that was introduced in [1] and [2] and symbol $N_{(k,j)}$ is by default the node at position j on level k of $\boldsymbol{T}_3$, where $k \ge 0$ and $0 \le j \le 2^k - 1$. By using the asterisk wildcard *, symbol $N_{(k,*)}$ means a node lying on level k. An integer X is said to be clamped on level k of $\boldsymbol{T}_3$ if $2^{k+1} \le X \le 2^{k+2}-1$ and symbol $X \,\square\, k$ indicates X is clamped on level k. If a positive integer X is clamped on level k and there is a node Y of $\boldsymbol{T}_3$ satisfying $X = \lfloor \sqrt{Y} \rfloor$, then X is said to be a floor square root of the node Y and Y is called a square source of X.

Remark 1. The concept that an integer is clamped on a level of $\boldsymbol{T}_3$ was first put forward in paper [3]. In the paper [3], a positive integer X was said to be clamped on level k of $\boldsymbol{T}_3$ if $2^{k+1}+1 \le X \le 2^{k+2}-1$. Since $2^{k+1}-1$ is the rightmost node on level $k-1$ and $2^{k+1}+1$ is the leftmost node on level k, there is an integer 2^{k+1} between the two. In order to avoid leaving out the number 2^{k+1}, this paper redefines it by $2^{k+1} \le X \le 2^{k+2}-1$.

2.2. Lemmas

Lemma 1 (See in [1]). T_3 Tree has the following fundamental properties.

(P1). Every node is an odd integer and every odd integer bigger than 1 must be on the T_3 tree. Odd integer N with $N > 1$ lies on level $\lfloor \log_2 N \rfloor - 1$.

(P2). On level k with $k = 0,1,...$, there are 2^k nodes starting by $2^{k+1}+1$ and ending by $2^{k+2}-1$, namely, $N_{(k,j)} \in [2^{k+1}+1, 2^{k+2}-1]$ with $j = 0,1,...,2^k - 1$.

(P3). $N_{(k,j)}$ is calculated by

$$N_{(k,j)} = 2^{k+1} + 1 + 2j, \; j = 0,1,...,2^k - 1$$

(P4) Multiplication of arbitrary two nodes of $\boldsymbol{T}_3$, say $N_{(m,\alpha)}$ and $N_{(n,\beta)}$, is a third node of $\boldsymbol{T}_3$. Let $J = 2^m(1+2\beta) \; +2^n(1+2\alpha) + 2\alpha\beta + \alpha + \beta$; the multiplication $N_{(m,\alpha)} \times N_{(n,\beta)}$ is given by

$$N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2} + 1 + 2J$$

If $J < 2^{m+n+1}$, then $N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+1,J)}$ lies on level $m+n+1$ of $\boldsymbol{T}_3$; whereas, if $J \geq 2^{m+n+1}$, $N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+2,\chi)}$ with $\chi = J - 2^{m+n+1}$ lies on level $m+n+2$ of $\boldsymbol{T}_3$.

Lemma 2 (See in [4]). For real numbers x, y and positive integer i, it holds

(P13) $x \leq y \Rightarrow \lfloor x \rfloor \leq \lfloor y \rfloor$; $x < n \Rightarrow \lfloor x \rfloor < n$, where n is an integer.

(P31) $i - 1 \leq 2\left\lfloor \frac{i}{2} \right\rfloor \leq i$.

3. Main Results

Theorem 1. Let k be a positive integer; then there are $2k+1$ consecutive integers $n_1, n_2, ..., n_{2k+1}$ that satisfy $\lfloor \sqrt{n_i} \rfloor_{(i=1,2,...,2k+1)} = k$.

Proof. Consider an arbitrary integer n such that $\lfloor \sqrt{n} \rfloor = k$; then by definition of the floor function it holds $k \leq \sqrt{n} < k+1$. That is

$$k^2 \leq n < k^2 + 2k + 1$$

Hence the $2k+1$ integers, $n_1 = k^2 + 0$, $n_2 = k^2 + 1, n_3 = k^2 + 2, ...$, and $n_{2k+1} = k^2 + 2k$, are the integers satisfying $\lfloor \sqrt{n_i} \rfloor_{(i=1,2,...,2k+1)} = k$.

Proposition 1. Let $N_{(m,\alpha)}$ be a node of $\boldsymbol{T}_3$ with $m > 0$; then when $0 \leq \alpha \leq \left\lfloor \frac{\sqrt{2^{2m+3}+1}-1}{2} \right\rfloor - 2^m$, $N^2_{(m,\alpha)}$ lies on level $2m+1$; otherwise it lies on level $2m+2$. Particularly, $N^2_{(m,0)} = N_{(2m+1,2^{m+1})}$ and $N^2_{(m,2^m-1)} = N_{(2m+2,2^{2m+2}-2^{m+2})}$.

Proof. Direct calculation shows

$$N^2_{(m,\alpha)} = (2^{m+1} + 2\alpha + 1)^2 = 2^{2m+2} + 2(2\alpha^2 + 2^{m+2}\alpha + 2^{m+1} + 2\alpha) + 1$$

By Lemma 1 (**P4** & **P5**), it knows that $N^2_{(m,\alpha)}$ lies on level $2m+1$ if and only if $J = 2\alpha^2 + 2^{m+2}\alpha + 2^{m+1} + 2\alpha < 2^{2m+1}$. Consequently,

$$\alpha^2 + 2^{m+1}\alpha + 2^m + \alpha < 2^{2m}$$
$$\Rightarrow \alpha^2 + (2^{m+1}+1)\alpha + 2^m - 2^{2m} < 0$$
$$\Rightarrow 0 \le \alpha < \frac{-(2^{m+1}+1) + \sqrt{(2^{m+1}+1)^2 - 4\times(2^m - 2^{2m})}}{2}$$
$$\Rightarrow 0 \le \alpha < \frac{\sqrt{2^{2m+3}+1} - (2^{m+1}+1)}{2}$$
$$\Rightarrow 0 \le \alpha < \frac{\sqrt{2^{2m+3}+1} - 1}{2} - 2^m$$
$$\Rightarrow 0 \le \alpha \le \left\lfloor \frac{\sqrt{2^{2m+3}+1} - 1}{2} \right\rfloor - 2^m$$

which validates the first part of the proposition.

The second part is easily obtained by the following calculations.

$$N_{(m,0)}^2 = (2^{m+1} + 2\times 0 + 1)^2 = 2^{2m+2} + 2\times 2^{m+1} + 1 = N_{(2m+1,2^{m+1})}$$

$$N_{(m,2^m-1)}^2 = (2^{m+1} + 2\times(2^m - 1) + 1)^2 = (2^{m+1} + 2^{m+1} - 1)^2$$
$$= 2^{2m+2} + 2^{2m+2} + 1 + 2^{2m+3} - 2^{m+2} - 2^{m+2}$$
$$= 2^{2m+3} + 2^{2m+3} - 2^{m+3} + 1$$
$$= 2^{2m+3} + 2\times(2^{2m+2} - 2^{m+2}) + 1$$
$$= N_{(2m+2,2^{2m+2}-2^{m+2})}$$

Remark 1. The condition $m > 0$ in Proposition 1 is proposed because it can get rid of the case $N_{(0,0)}^2 = 3^2 = 9 = N_{(2,0)}$, which is the unique example that violates $N_{(m,0)}^2 = N_{(2m+1,2^{m+1})}$.

Proposition 2. Let k be a positive integer, $N_{(2k+1,*)}$ and $N_{(2k+2,*)}$ be nodes on of $\boldsymbol{T}_3$; then $(\lfloor \sqrt{N_{(2k+1,*)}} \rfloor \le \lfloor 2^{k+1}\sqrt{2} \rfloor) □ k$ and $(\lfloor 2^{k+1}\sqrt{2} \rfloor \le \lfloor \sqrt{N_{(2k+2,*)}} \rfloor) □ k$. On level $2k$ there is not a node $N_{(2k,*)}$ satisfying $N_{(2k,*)} □ k$ and there is neither a node $N_{(2k+3,*)}$ satisfying $N_{(2k+3,*)} □ k$ on level $2k+3$.

Proof. $N_{(2k+1,*)}$ and $N_{(2k+2,*)}$ being the nodes on levels $2k+1$ and $2k+2$ respectively yields

$$2^{2k+2} < 2^{2k+2} + 1 \le N_{(2k+1,*)} \le 2^{2k+3} - 1 < 2^{2k+3}$$
$$\Rightarrow 2^{k+1} = \lfloor \sqrt{2^{2k+2}} \rfloor \le N_{(2k+1,*)} \le \lfloor 2^{k+1}\sqrt{2} \rfloor$$

and

$$2^{2k+3} < 2^{2k+3} + 1 \le N_{(2k+2,*)} \le 2^{2k+4} - 1 < 2^{2k+4}$$
$$\Rightarrow \lfloor 2^{k+1}\sqrt{2} \rfloor \le \lfloor \sqrt{N_{(2k+2,*)}} \rfloor < 2^{k+2}$$

Note that $1 + \frac{1}{2} - \frac{1}{4} < \sqrt{2} < 1 + \frac{1}{2}$, it yields

$$2^{k+1} + 2^k - 2^{k-1} \le \lfloor 2^{k+1}\sqrt{2} \rfloor \le \left(\left\lfloor 2^{k+1}(1 + \frac{1}{2}) \right\rfloor \right) = 2^{k+1} + 2^k$$

Hence it holds

$$2^{k+1} \le N_{(2k+1,*)} \le \lfloor 2^{k+1}\sqrt{2} \rfloor \le 2^{k+1} + 2^k$$

and

$$2^{k+1} + 2^k - 2^{k-1} \le \lfloor 2^{k+1}\sqrt{2} \rfloor \le \lfloor \sqrt{N_{(2k+2,*)}} \rfloor < 2^{k+2}$$

That is $(\lfloor\sqrt{N_{(2k+1,*)}}\rfloor \le \lfloor 2^{k+1}\sqrt{2}\rfloor) \square k$ and $(\lfloor 2^{k+1}\sqrt{2}\rfloor \le \lfloor\sqrt{N_{(2k+2,*)}}\rfloor) \square k$.

Considering the biggest node $N_{(2k,2^{2k}-1)}$ on level $2k$, it yields

$$N_{(2k,*)} \le N_{(2k,2^{2k}-1)} = 2^{2k+2}-1 < 2^{2k+2} \Rightarrow \lfloor\sqrt{N_{(2k,*)}}\rfloor \le \lfloor\sqrt{N_{(2k,2^{2k}-1)}}\rfloor < 2^{k+1}$$

Likewise, considering the smallest node $N_{(2k+3,0)}$ on level $2k+3$, it yields

$$N_{(2k+3,0)} = 2^{2k+4}+1 > 2^{2k+4} \Rightarrow \lfloor\sqrt{N_{(2k+3,*)}}\rfloor \ge \lfloor\sqrt{N_{(2k+3,0)}}\rfloor \ge 2^{k+2} > 2^{k+2}-1$$

Hence there is not a node $N_{(2k,*)}$ satisfying $N_{(2k,*)} \square k$, and there is neither a node $N_{(2k+3,*)}$ satisfying $N_{(2k+3,*)} \square k$.

Proposition 3. Node $N_{(k,j)}$ $(k>1)$ of T_3 satisfies

$$2^{\lfloor\frac{k+1}{2}\rfloor}-1 < 2^{\lfloor\frac{k+1}{2}\rfloor} \le \lfloor\sqrt{N_{(k,j)}}\rfloor \le 2^{\lfloor\frac{k+1}{2}\rfloor+1}-1 < 2^{\lfloor\frac{k+1}{2}\rfloor+1}$$

Or equivalently $\lfloor\sqrt{N_{(k,j)}}\rfloor \square \left\lfloor\frac{k-1}{2}\right\rfloor$.

Proof. Since $2^{k+1}+1 \le N_{(k,j)} \le 2^{k+2}-1$, it yields $2^{k+1} < N_{(k,j)} < 2^{k+2}$; hence it holds

$$2^{\frac{k+1}{2}} < \sqrt{N_{(k,j)}} < 2^{\frac{k}{2}+1}$$

By Lemma 2(**P31**), the inequality $k \le 2\left\lfloor\frac{k+1}{2}\right\rfloor \le k+1$ yields $2^{\lfloor\frac{k+1}{2}\rfloor} \le 2^{\frac{k+1}{2}}$ and $2^{\frac{k}{2}+1} \le 2^{\lfloor\frac{k+1}{2}\rfloor+1}$, hence it holds

$$2^{\lfloor\frac{k+1}{2}\rfloor} < \sqrt{N_{(k,j)}} < 2^{\lfloor\frac{k+1}{2}\rfloor+1}$$

By Lemma 2 (**P13**) it immediately leads to

$$2^{\lfloor\frac{k+1}{2}\rfloor} \le \lfloor\sqrt{N_{(k,j)}}\rfloor < 2^{\lfloor\frac{k+1}{2}\rfloor+1}$$

which is $\lfloor\sqrt{N_{(k,j)}}\rfloor \square (\left\lfloor\frac{k+1}{2}\right\rfloor - 1 = \left\lfloor\frac{k-1}{2}\right\rfloor)$.

Remark 2. This proposition is a modification of the Corollary 2 in paper [3]. In paper [3], it claimed that $\lfloor\sqrt{N_{(k,j)}}\rfloor \square \left\lfloor\frac{k+1}{2}\right\rfloor - 1$ or $\lfloor\sqrt{N_{(k,j)}}\rfloor \square \left\lfloor\frac{k}{2}\right\rfloor$. However, seeing from Propositions 1, 2 and 3, one can see that $\lfloor\sqrt{N_{(k,j)}}\rfloor \square \left\lfloor\frac{k+1}{2}\right\rfloor$ never occurs.

Theorem 2. Given $N>3$ be an odd integer; then $\lfloor\sqrt{N}\rfloor \square \left\lfloor\frac{\lfloor\log_2 N\rfloor}{2}\right\rfloor - 1$.

Proof. Let $k = \lfloor\log_2 N\rfloor - 1$. By Lemma 1(**P1**), N is a node on level k of $\boldsymbol{T}_3$. By Proposition 3 it knows $\lfloor\sqrt{N}\rfloor \square \left\lfloor\frac{k-1}{2}\right\rfloor$, that is $\lfloor\sqrt{N}\rfloor \square \left\lfloor\frac{\lfloor\log_2 N\rfloor}{2}\right\rfloor - 1$.

Example 1. Table 1 lists several odd integers that are randomly picked, and their positions in $\boldsymbol{T}_3$ as well as their square roots in $\boldsymbol{T}_3$. It can see that, $\lfloor\sqrt{N}\rfloor \square \left\lfloor\frac{\lfloor\log_2 N\rfloor}{2}\right\rfloor - 1$ holds for each number. Readers can check it manually or with Mathematica.

Table1. *Odd Integers and its square roots in* $\boldsymbol{T}_3$

Odd Integer N	N's in $\boldsymbol{T}_3$	$\left\lfloor \frac{\lfloor \log_2 N \rfloor}{2} \right\rfloor - 1$	$\lfloor \sqrt{N} \rfloor$ & its level
1517	$N_{(9,246)}$	4	$\lfloor \sqrt{1517} \rfloor = 38 \square 4$
20491	$N_{(13,2053)}$	6	$\lfloor \sqrt{20491} \rfloor = 143 \square 6$
386757	$N_{(17,62306)}$	8	$\lfloor \sqrt{386757} \rfloor = 621 \square 8$
6947533	$N_{(21,1376614)}$	10	$\lfloor \sqrt{6947533} \rfloor = 2635 \square 10$
104678919	$N_{(25,18785027)}$	12	$\lfloor \sqrt{104678919} \rfloor = 10231 \square 12$

Remark 3. Table 1 can be easily checked manually or with Mathematica. When programmed as follows,

```
f[x_]:=Floor[Floor[Log[x]/Log[2]]/2]-1;
g[x_]:=Floor[Log[Sqrt[x]]/Log[2]]-1;
r[x_]:=Floor[Sqrt[x]];
inData={1517,20491,386757,6947533,104678919};
r1=Table[f[inData[[i]]],{i,5}];
r2=Table[g[inData[[i]]],{i,5}];
r3=Table[r[inData[[i]]],{i,5}];
t={r1,r2,r3}//MatrixForm
```

the screenshot in Mathematica 7.0 is as figure 1

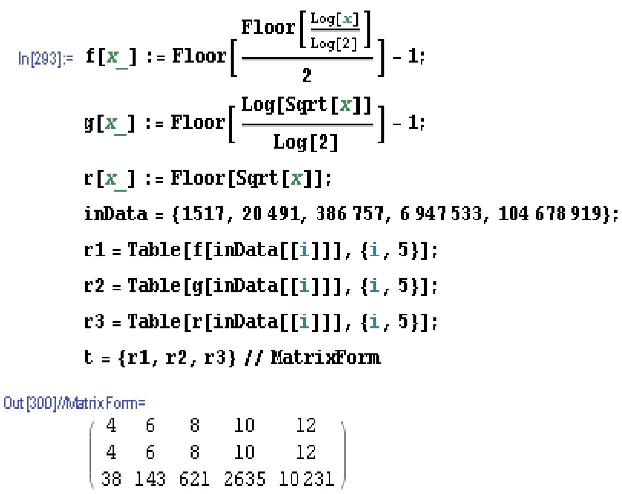

Fig1. *Screenshot of the program and output*

Corollary 1. Let $N > 3$ be an odd integer and $k = \lfloor \log_2 N \rfloor - 1$; then $\left(\lfloor \sqrt{N} \rfloor \le \left\lfloor 2^{\left\lfloor \frac{k+1}{2} \right\rfloor} \sqrt{2} \right\rfloor \right) \square \left\lfloor \frac{k-1}{2} \right\rfloor$ when k is odd whereas $\left(\left\lfloor 2^{\left\lfloor \frac{k+1}{2} \right\rfloor} \sqrt{2} \right\rfloor \le \lfloor \sqrt{N} \rfloor \right) \square \left\lfloor \frac{k-1}{2} \right\rfloor$ when k is even.

Proof. By Theorem 2, $\lfloor \sqrt{N} \rfloor \square \left\lfloor \frac{k-1}{2} \right\rfloor$ is sure. Now consider the fact that, whether $k = 2l+1$ or $k = 2l+2$, $\left\lfloor \frac{k+1}{2} \right\rfloor = l+1$ always holds. By Proposition 2, it knows that, $\left(\lfloor \sqrt{N} \rfloor \le \left\lfloor 2^{\left\lfloor \frac{k+1}{2} \right\rfloor} \sqrt{2} \right\rfloor \right) \square \left\lfloor \frac{k-1}{2} \right\rfloor$ when k is odd whereas $\left(\left\lfloor 2^{\left\lfloor \frac{k+1}{2} \right\rfloor} \sqrt{2} \right\rfloor \le \lfloor \sqrt{N} \rfloor \right) \square \left\lfloor \frac{k-1}{2} \right\rfloor$ when k is even.

4. CONCLUSION

The square and the square root are important numbers. For an integer, the square root is essentially important because it is the cut-off point of divisors of a composite integer. Study of these numbers is helpful to understand distribution of the divisors on T_3. The Theorem 2 proved in this paper is of course a foundation for us to know where the square root of a node lies. Hope it is helpful in the future.

ACKNOWLEDGEMENTS

The research work is supported by the State Key Laboratory of Mathematical Engineering and Advanced Computing under Open Project Program No.2017A01, Department of Guangdong Science and Technology under project 2015A010104011, Foshan Bureau of Science and Technology under projects 2016AG100311, Project gg040981 from Foshan University. The author sincerely present thanks to them all.

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WANG Xingbo, was born in Hubei, China. He got his Master and Doctor's degree at National University of Defense Technology of China and had been a staff in charge of researching and developing CAD/CAM/NC technologies in the university. Since 2010, he has been a professor in Foshan University, still in charge of researching and developing CAD/CAM/NC technologies. Wang has published 8 books, over 100 papers and obtained more than 30 patents in mechanical engineering

Citation: *WANG, X. (2018). More on Square and Square Root of a Node on T3 Tree. International Journal of Scientific and Innovative Mathematical Research (IJSIMR), 6(5), pp.45-50. http://dx.doi.org/ 10.20431 /2347-3142.0605005*