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Superiority of Graph Theoretic Approach to Vogel's Approximation Method in Solving Unbalanced Transportation Problem

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Abstract: *In this work, we use Graph Theoretic Approach to solve unbalanced transportation problem which is a special class of Linear Programming Problem. We represented the transportation problem as a bipartite graph and solved it iteratively. To illustrate the method, two numerical examples are solved and the obtained solutions are compared with those obtained via Vogel approximation method. The present method yields superior initial basic feasible solution of the problems. Moreover, the present method is found to be very easy to understand; use and implement compared to the Vogel approximation method and can be applied on real life transportation problems by the decision makers.*

Keywords: *Optimization, Graph, Transportation, Spanning Tree, Network Flow.*

1. INTRODUCTION

Transportation Problem (TP) is an optimization problem common to business organizations and it is a particular class of linear programming, associated with day-to-day activities in our real life. The concept is primarily concerned with the optimal (best possible) way in which a product produced at different factories or plants (called supply origins) can be transported to a number of warehouses, wholesalers, distributors or customers (called demand destinations). For instance, a firm produces goods at $m$different supply centers, labeled as $S_1, S_2, \dots, S_m$. The demand for the goods is spread out at n different demand centers labeled $D_1, D_2, \dots, D_n$. The problem of the firm is to get goods from supply centers to demand centers at minimum cost in such a way that the total supplies from all Sources are sufficient to meet the demands at all the destinations. The transportation problem is an important linear programming model that arises in several contexts and has deservedly received much attention in literature. It was first formulated by Hitchcock (1941), and was independently treated by Koopmans (1949). However, the problem was solved for optimum solutions as answers to complex business problems when Dantzig (1963) applied the concept of Linear programming in solving the transportation model using the simplex method. Since then the transportation problem has become the classical common subject in almost every textbook on operation research and mathematical programming. The common methods used to obtain an initial feasible solution of the problem include: the Northwest-corner method (NWCM), Least-cost method (LCM) or Vogel approximation method (VAM). Finding an initial basic feasible solution using Vogel approximation method was the work of Renifeld and Vogel, (1958). In general, the Vogel's approximation method usually tends to produce an optimal or near optimal initial solution. Several researches in this field determined that Vogel produces an optimum solution in about 80% of the problems under test, according to Rekha (2013) and Vivek (2009). However, the method is hard to comprehend, highly technical to handle and time consuming, involves high cost of implementation, etc. Many people propose modifications to VAM for obtaining initial solutions to the unbalanced transportation problem. Shimshak *et al*. (1981) propose a modification (SVAM) which ignores any penalty that involves a dummy row/column.

Goyal (1984) suggests another modification in (GVAM) where the cost of transporting goods to or from a dummy point is set equal to the highest transportation cost in the problem, rather than to zero. The method proposed by Ramakrishnan (1988) which was an improvement on Goyal's work consists of four steps of reduction and one step of VAM. Edward and Raja (2016) used Advanced Approximation Method (AAM) for solving an unbalanced fuzzy transportation problem. The method produced an optimal solution, without converting into a balanced one as is the case with other methods. Aljanabi and Jasim (2015) proposed a new approach for solving transportation problem using modified kruskal algorithm. The approach was bias to graph theory while being supported by the kruskal algorithm for finding minimum spanning tree.

1.1. Statement of the Problem

Description of a classical transportation problem can be given as follows: A certain amount of homogeneous commodity is available at a number of sources and a fixed amount is required to meet the demand at a number of destinations. Then finding a method that is simple, easy to understand and cost effective for use in obtaining an optimal schedule of shipment of the commodity with the satisfaction of demands at each destination is the main goal of the problem.

1.2. Aim

The aim of this work is to establish the superiority of the graph theoretic approach to Vogel's approximation method in solving unbalanced transportation problem.

1.3. Objectives

i. To represent the unbalanced transportation problem with bipartite graph.

ii. To obtain an initial feasible solution of the problem using graph contraction method.

iii. To verify the scalability of the method.

2. METHODS

2.1. Mathematical Formulation

Let us consider the m-plant locations (sources) as $S_1, S_2, \dots S_m$ and the n-retail Depots (destination) as $D_1, D_2, \dots D_n$ respectively. Let $a_i \geq 0, i = 1,2,3, \dots m$, be the amount available at plant S_i. Let the amount required at depot D_j be $b_j \geq 0, j = 1,2,3, \dots n$. Let the cost of transporting one unit of product from S_i to D_j be C_{ij}, $i = 1,2, \dots m, j = 1,2,3, \dots n$ If $x_{ij} \geq 0$ is the number of units to be transported from S_i to D_j, then the problem is to determine x_{ij} so as to minimize the objective function (z)

$$Min \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} C_{ij}\, x_{ij} \qquad (3.1)$$

Formula (3.1) is subjected to two constraints namely:

Supply constraints:

$$\sum_{j=1}^{j=n} x_{ij} \leq a_i \; (i = 1,2, \dots, m) \qquad (3.2)$$

Demand constraints:

$$\sum_{i=1}^{i=m} x_{ij} = b_j \; (j = 1,2, \dots, n) \qquad (3.3)$$

Since all the x_{ij} must be non-negative, the above equations need to satisfy the following additional restriction called the non-negativity constraint:

thus, $x_{ij} \geq 0 \; (i = 1,2, \dots, m; j = 1,2, \dots, n)$

Remark: The set of constraints represents $m + n$ equations in m x n non-negative variables.

This problem can be represented by a Table as shown in Table 1.

Table1.*A typical Transportation Problem Tableau*

To / From	D_1	D_2	. . .	D_n	Supply a_i
S_1	C_{11} x_{11}	C_{12} x_{12}	. . .	C_{1n} x_{1n}	a_1
S_2	C_{21} x_{21}	C_{22} x_{22}	. . .	C_{2n} x_{2n}	a_2
. .	. .	. .	. . . . . .	. .	. .
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}	. . .	C_{mn} x_{mn}	a_m
b_j	b_1	b_2	. . .	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Source: *(Aljanabi and Jasim, 2015).*

The graph that appears in the transportation problem is called bipartite graph, or a graph in which all vertices can be divided into two classes, so that no edge connects two vertices in the same class. A property of bipartite graphs is that they are 2-colourable, that is, it is possible to assign each vertex a "colour" out of a 2-element set so that no two vertices of the same colour are connected. We apply our knowledge of spanning trees, network flows and matching to study minimum-cost network flow. To simplify the problem, consider networks whose underlying graphs are complete bipartite. All edges go directly from suppliers to demands and have finite capacity and known per unit cost. We refer to the supply vertices as Sources and the demand as Destinations and edge (i,j) as the link from each sourceS_i to each destinationD_j.There is a cost$C(i,j)$, the unit cost, charged for shipping an item on edge(i,j). The goal is to find a routing of all the items from Sources to destinations that minimizes the transportation costs. This optimization problem is appropriately called the Transportation Problem. It is one of the first optimization problems studied in operations research. In general, warehouse i has a supply of size $S(i)$ and store$j$ has a demandof size $D(j)$. We assume that the total supplied summed over all warehouses, are adequate to meet the demand at all the stores. A solution to the transportation problem specifies the amount $X(i;j)$to ship the goods from each sourceS_i to each destination D_j which is minimum.

2.2. Network Representation of the Problem as Bipartite Graph

The network representation of the problem is given as Figure 1.

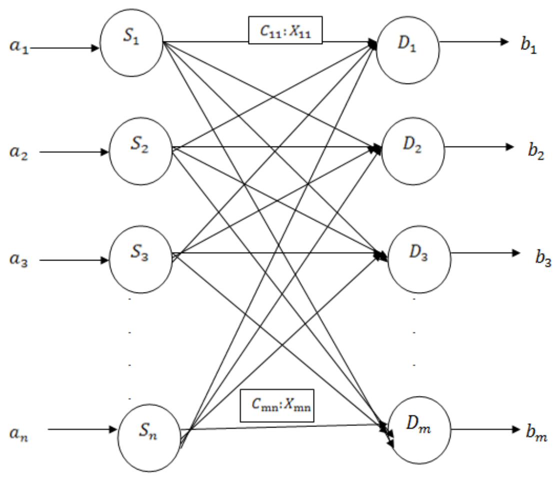

Figure1. *Graphical Representation of a Typical Transportation Problem*

Source: *(Hamdy, 2013)*

2.3. Unbalanced Transportation Problems

A typical case of a transportation problem in reality most often does not appear balanced, i.e., the total number of goods demanded does not always equal the total number of goods supplied ($\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$).

Such form of transportation problem is called the unbalanced transportation problem, where equation (3) amongst the afore-listed constraints set is violated. Such violation of the balance constraint presents itself in either of two different cases as we shall see in the following:

i. $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, this case is where we have the total supply is exceeding the total demand. To correct this, dummy variables are added to the demand constraint to cover up the deficiency; and the difference $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ becomes the summed up demand for that dummy variable.

ii. $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$, this second case of the unbalanced type of the transportation problem is the case where total supply is exceeded by total demand by some amount. To correct this, once again, we introduce dummy variables; and the difference $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$ becomes the capacity for the dummy supply centre.

In operations research, obtaining the optimum solution of the transportation problem is achieved in two phases. The first phase is concerned with finding a basic feasible solution. This is a solution which satisfies all the constraints (including the non-negativity constraint) present in the problem. Different techniques and algorithms are used such as:

i. Northwest-corner method,

ii. Least-cost method and

iii. Vogel approximation method.

The present method depends on graph theoretic approach namely, graph contraction technique. Graph contraction is a technique for implementing recursive graph algorithms, where on each iteration; the algorithm is repeated on a smaller graph contracted from the previous step.

2.4. Algorithm for Graph Contraction Technique

The method consists of the following steps:

Step 1: Balance the Transportation problem by introducing a dummy and set the cost of transporting goods to or from it equal to the highest transportation cost in the problem, rather than to zero.

Step 2: Convert the Transportation Tableau into Bipartite Graph.

Step 3: Select the edge with the least unit cost.

Step 4: Assign as much allocation as possible to this edge depending on quantities at source and demand whichever is less.

Step 5: Delete the source or demand vertex depending on which one is satisfied first.

Step 6: Repeat from step 2 until there is no edge left.

3. RESULTS

The following examples are considered to study the correctness, effectiveness and scalability of the present technique.

3.1. Example 1

Consider the Transportation problem of a company that has three factories (S_i) $i = 1, 2, 3$ supplying three warehouses (D_j) $j = 1, 2, 3, 4$ and its management wants to determine the minimum-cost shipping schedule for its annual output. Factory supply, depot demands, and shipping costs per unit of product are shown in Table 2 below.

Table2. *Data for Transportation Problem Example 1*

Destination Source	D_1 ($)	D_2 ($)	D_3 ($)	D_4 ($)	Supply (a_i)
S_1	270	230	310	690	100
S_2	100	450	400	320	80
S_3	300	540	350	570	80
S_4	690	690	690	690	(10)
Demand (b_j)	60	120	50	40	$\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

Source: *(Vivek , 2009)*

Let X_{ij} = number of tons of cement transported from i^{th} Source $(i = 1,2,3)$ to j^{th} Destination, $(j = 1, 2, 3, 4)$ implying that $m = 3$ and $n = 4$.

The problem can be formulated mathematically in the linear programming form as:

Minimize $Z = 270x_{11} + 230x_{12} + 310x_{13} + 690x_{14} + 100x_{21} + 450x_{22} + 400x_{23} + 320x_{24} + 300x_{31} + 540x_{32} + 350x_{33} + 570x_{34} + 690x_{41} + 690x_{42} + 690x_{43} + 690x_{44}$

Subject to

Capacity constraint:

$$270x_{11} + 230x_{12} + 310x_{13} + 690x_{14} \leq 100$$

$$100x_{21} + 450x_{22} + 400x_{23} + 320x_{24} \leq 80$$

$$300x_{31} + 540x_{32} + 350x_{33} + 570x_{34} \leq 80$$

$$690x_{41} + 690x_{42} + 690x_{43} + 690x_{44} \leq 10$$

Requirement constraint:

$$270x_{11} + 100x_{21} + 300x_{31} + 690x_{41} \leq 60$$

$$230x_{12} + 450x_{22} + 540x_{32} + 690x_{42} \leq 120$$

$$310x_{13} + 400x_{23} + 350x_{33} + 690x_{43} \leq 50$$

$$690x_{14} + 320x_{24} + 570x_{34} + 690x_{44} \leq 40$$

Non-negativity constraint:

$x_{ij} \geq 0$

The above problem has 8 constraint equations and 16 decision variables (i.e. $(n+m)$ *and* $(n \text{ x } m)$ respectively). The network representation of the problem as a bipartite graph is as shown in Figure 2 below.

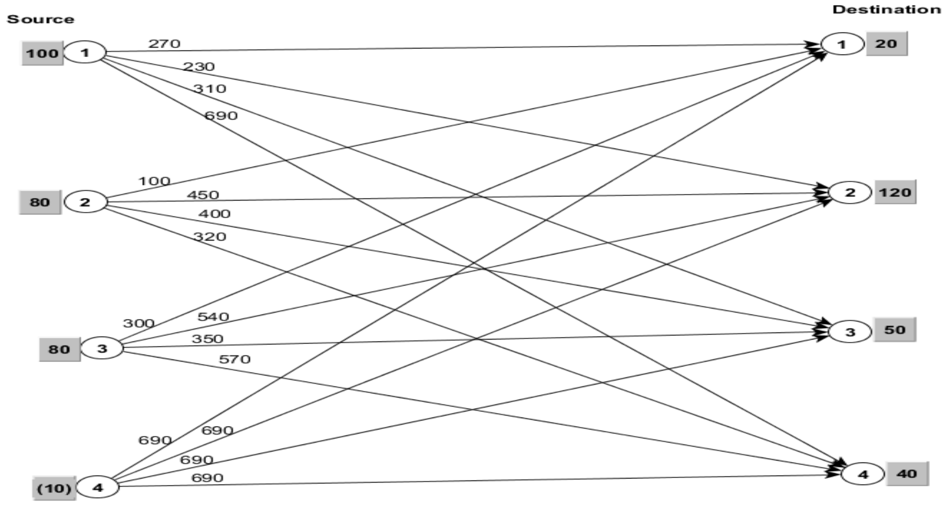

Figure2. *Graph of Problem Example 1*

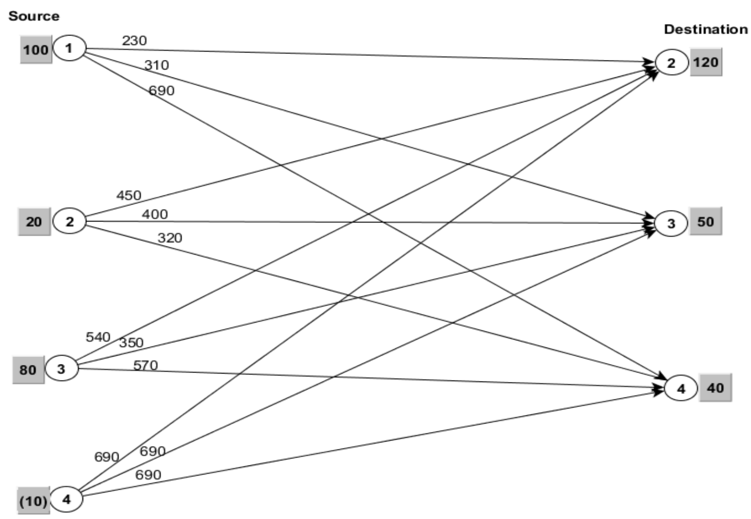

Figure3. *Graph of Iteration 1 (after allocating 60 units of the product to edge S_2D_1 at 100 unit of cost, i.e. 6,000)*

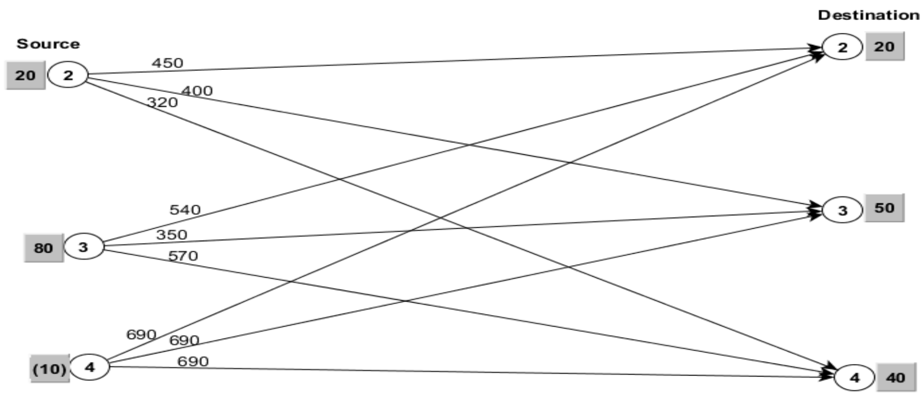

Figure4. *Graph of Iteration 2 (after allocating 100 units of the product to edge S_1D_2 at 230 unit of cost, i.e. 23,000)*

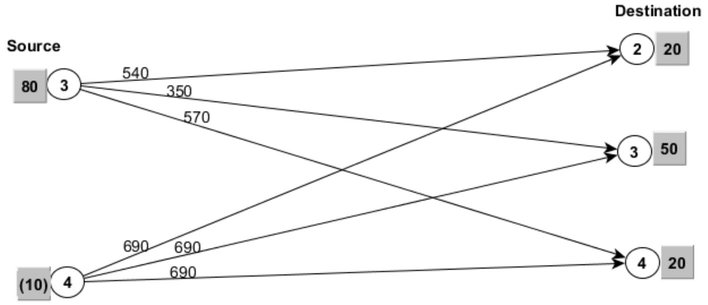

Figure5. *Graph of Iteration 3 (after allocating 320 units of the product to edge* S_2D_4 *at* 20 *unit of cost, i.e.* $6{,}400$*)*

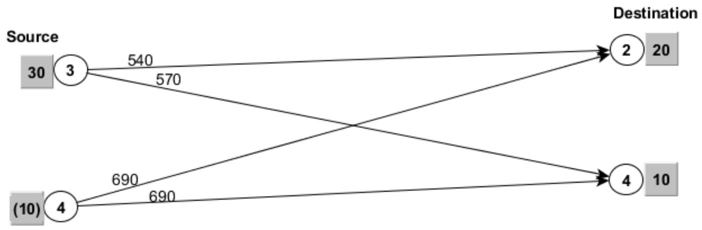

Figure6. *Graph of Iteration 4 (after allocating 50 units of the product to edge* S_3D_3 *at* 350 *unit of cost, i.e.* $17{,}500$*)*

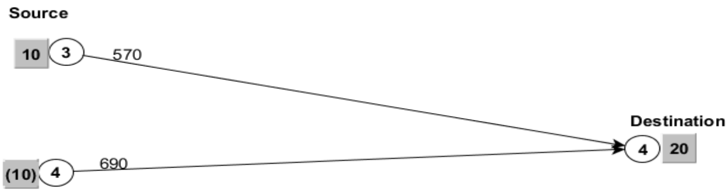

Figure7. *Graph of Iteration 5 (after allocating 20 units of the product to edge* S_1D_2 *at* 540 *unit of cost, i.e.* $10{,}800$*)*

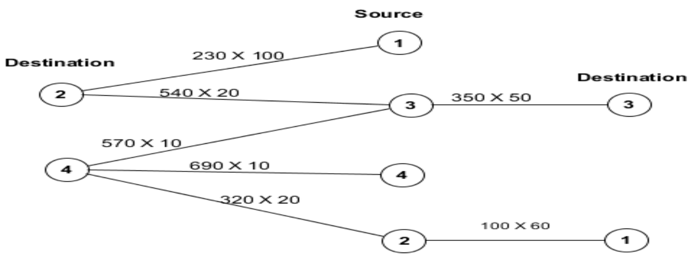

Figure8. *Minimum Spanning Tree of Problem Example 1*

From Figure 8 we have that

$$x_{11} = 0,\quad x_{12} = 100,\quad x_{13} = 0,\quad x_{14} = 0,\quad x_{21} = 60,\quad x_{22} = 0,\quad x_{23} = 0,\quad x_{24} = 20,\quad x_{31} = 0,\quad x_{32} = 20,\quad x_{33} = 50,\quad x_{34} = 10,\quad x_{41} = 0,\quad x_{42} = 0,\quad x_{43} = 0,\quad x_{44} = 10$$

Thus,

$$Z = 230x_{12} + 100x_{21} + 320x_{24} + 540x_{32} + 350x_{33} + 570x_{34} + 690x_{44}$$

$=$ 23,000 + 6,000 + 6,400 + 10,800 + 17,500+ 5,700 + (6,900)

$=$ 62,500

Table3. *Result of Problem Example 1*

S/N	Edge	Cost ($)	(Qty)	Amount ($)
1	S_2D_1	100	60	6,000
2	S_1D_2	230	100	23,000
3	S_2D_4	320	20	6,400
4	S_3D_3	350	50	17,500
5	S_3D_2	540	20	10,800
6	S_3D_4	570	10	5,700
7	S_4D_4	690	10	(6,900)
Total Cost				62,500

3.2. Example 2

Consider the Transportation problem of a company that has three plants $(S_i)\ i = 1,2,3$ supplying three key distributors $(D_j)\ j = 1,2,3$ and its management wants to determine the best shipping schedule for its annual output. Factory supply, depot demands, and shipping costs per unit of product are shown in Table 10 below.

Table4. *Data for Transportation Problem Example 2*

Destination Source	D_1	D_2	D_3	D_4	Supply (a_i)
S_1	4	8	8	24	76
S_2	16	24	16	24	82
S_3	8	16	21	24	77
Demand (b_j)	72	102	41	(20)	$\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

Source: (*Nigus and Tripti, 2013)*

Letx_{ij} = number of units of the produced good transported from i^{th} Source $(i = 1,2,3)$ to j^{th} Destination, $(\text{j} = 1,2,3)$ implying that $m = 3$ and $n = 3$.

The problem can be formulated mathematically in the linear programming form as:

Minimize $Z = 4x_{11} + 8x_{12} + 8x_{13} + 24x_{14} + 16x_{21} + 24x_{22} + 16x_{23} + 24x_{24} + 8x_{31} + 16x_{32} + 21x_{33} + 24x_{34}$

Subject to

Capacity constraint:

$$4x_{11} + 8x_{12} + 8x_{13} + 24x_{14} \leq 76$$

$$16x_{21} + 24x_{22} + 16x_{23} + 24x_{24} \leq 82$$

$$8x_{31} + 16x_{32} + 21x_{33} + 24x_{34} \leq 77$$

Requirement constraint:

$$4x_{11} + 16x_{21} + 8x_{31} \leq 72$$

$$8x_{12} + 24x_{22} + 16x_{32} \leq 102$$

$$8x_{13} + 16x_{23} + 21x_{33} \leq 41$$

$$24x_{14} + 24x_{24} + 24x_{34} \leq (20)$$

Non-negativity constraint:

$x_{ij} \geq 0$

The above problem has 7 constraint equations and 12 decision variables (i.e. $(n + m)$ *and* $(n \text{ x } m)$ respectively).

The network representation of the problem as a bipartite graph is as in Figure 8 below.

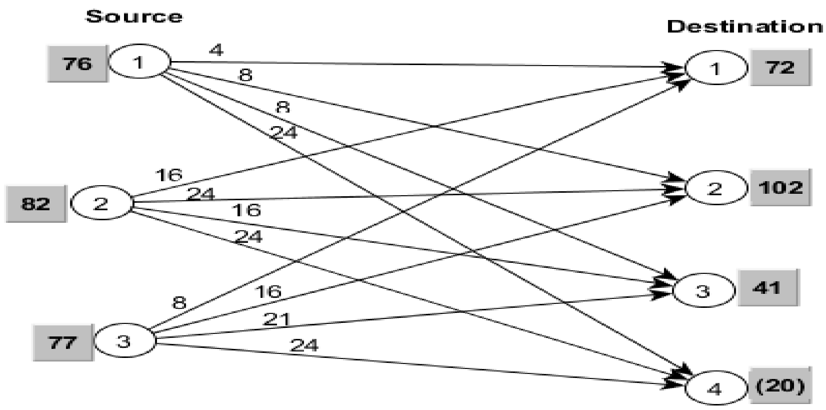

Figure9. *Graph of Problem Example 2*

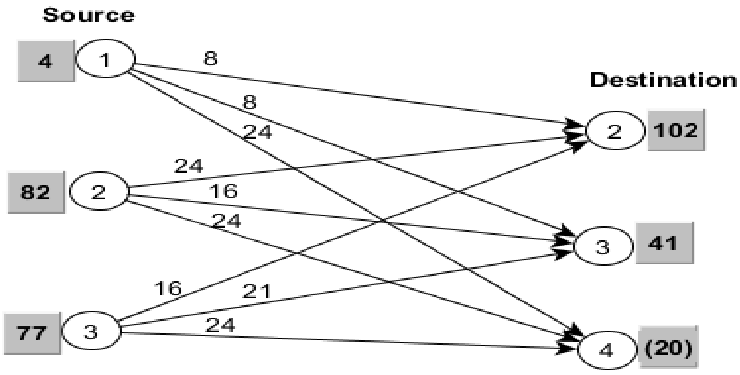

Figure10. *Graph of Iteration 1 (after allocating 72 units of the produced good to edge* S_1D_1 *at* 4 *unit of cost, i.e.* 288*)*

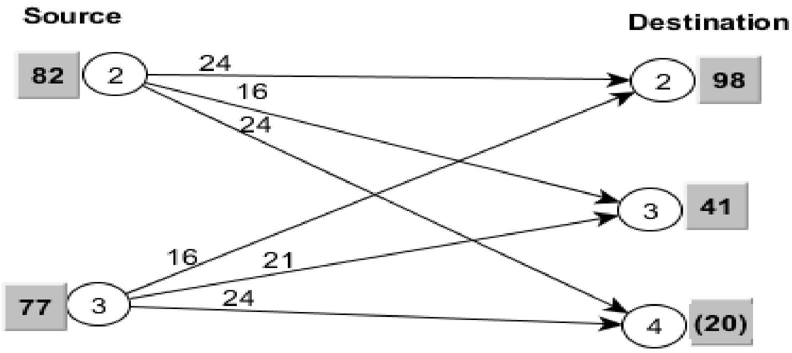

Figure11. *Graph of Iteration 2 (after allocating 4 units of the produced good to edge* S_1D_2 *at* 8 *unit of cost, i.e.* 32*)*

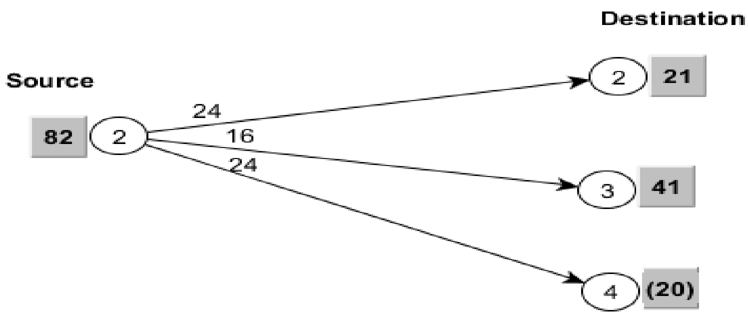

Figure12. *Graph of Iteration 3 (after allocating 77 units of the produced good to edge* S_3D_2 *at* 16 *unit of cost, i.e.* $1,232$*)*

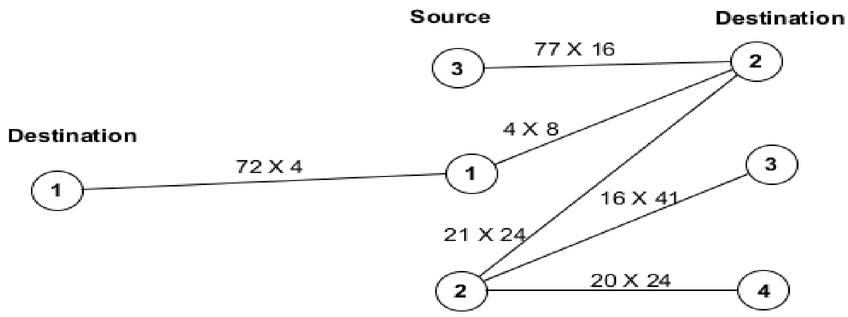

Figure13. *Minimum Spanning Tree of Problem Example 2*

From Figure 13 we have that

$$x_{11} = 72, \quad x_{12} = 4, \quad x_{13} = 0, \quad x_{14} = 0, \quad x_{21} = 0, \quad x_{22} = 21, \quad x_{23} = 41, \quad x_{24} = 20, \quad x_{31} = 0, \quad x_{32} = 77, \quad x_{33} = 0, \quad x_{34} = 0$$

Thus objective function,

$$Z = 4x_{11} + 8x_{12} + 24x_{22} + 16x_{23} + 24x_{24} + 16x_{32}$$

= 288 + 32 + 1,232 + 504 + 656 + (480)

= 2,232

Table5. *Result of Problem Example 2*

S/N	Edge	Cost ($)	(Qty)	Amount
1	S_1D_1	4	72	288
2	S_1D_2	8	4	32
3	S_3D_2	16	77	1, 232
4	S_2D_2	24	21	504
5	S_2D_3	41	16	656
6	S_2D_4	20	24	(480)
Total Cost				2,232

3.3. Tabulation of Results

To illustrate the comparative efficiency, the solutions obtained using Vogel's Approximation Method (VAM) by the two researchers and the present method, Graph Contraction Technique (GCT), are shown in Table 6 below.

Table6.*Summary of Solutions Obtained by Both Methods*

Problem	Source	Solution byExample	Solution by Example
		(VAM)	(GCT)
1.	Vivek (2009)	68,900	62,500
2.	Nigus and Tripti (2013)	2,524	2,232

4. Discussion

The presented approach for finding the initial basic feasible solution of the transportation problem is based mainly on using graph contraction technique (GTC) for determining the most cost effective shipment schedule.

From the results given in Table 6 above, it is clear that the technique can be used in different transportation models and gives faster convergence criteria since it is based mainly on reduction of the graph after each iteration.

5. Conclusion

In this thesis we have used a graph theoretic approach to solve unbalanced transportation problem that have earlier been solved by different researchers using Vogel's Approximation Method (VAM).

From the investigations and the results given above, we found that the Present Method givessuperior results for all unbalanced transportation problems has the merit that it produces an optimal solution in over eighty percent of all cases. Moreover, the method presented here is simpler in comparison with Vogel's Approximation Method (VAM). It is very easy to understand, involves simple calculations and thus saves time and can be easily applied to find the initial basic feasible solution for the any unbalanced transportation problems.

RECOMMENDATION

The transportation problem remains a major problem in distribution firms, and as such should be effectively given managerial concern. In view of the importance and relevance of this problem, new results in this area are also important, especially if they can be used for practical purposes in solving problems of greater scope.

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