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Some Results about the Special Partial Ordering

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Abstract: *The Lowner partial ordering of semi-positive definite matrices is an important matrix partial order relationship. In this paper, we will give some Proofs about the properties of the Lowner partialordering, including the results about relationship between semi-positive and its squared matrix, Groβ prove it.*

Keywords: *Lowner partial ordering Lowner - Heniez*

1. Introduction

The *Lowner* partial ordering was proposed by the German mathematician $L\ddot{o}wner$ in 1934. After that, many mathematicians studied the nature and characterization of this partial order, and it has been widely used in statistics. In 1989, Bakalary studied the order of the matrix and the $L\ddot{o}wner$ partial ordering, and got some better results.

For semi-positive definite matrices, Baksarary discusses the relationship between the positive definite matrix and the $L\ddot{o}wner$ partial ordering of its square matrix, and the following conclusions are obtained. $A^2 \le B^2$, $AB = BA \Rightarrow A \le B$.

For the $L\ddot{o}wner$ partial ordering of a semi-positive definite matrix, $Gr\ddot{o}\beta$ gives the following conclusions. $A \le B \Leftrightarrow A = BK$, $\lambda(K) \subseteq [0,1]$, $A^2 \le B^2 \Leftrightarrow A = BK$, $\lambda(KK^*) \subseteq [0,1]$.

2. Some Discussions About the *Lowner* Partial Ordering

The objects discussed in this paper are all semi-positive matrix on the real number field. If the complex field is involved, it will be displayed in the proposition. $A \ge B$ indicate that $A - B$ is a semi-positive matrix, $\lambda_{\max} A$ represents the maximum eigenvalue of matrix A, $\delta_{\max} A$ represents the square root of the largest eigenvalue of the product of A and its transpose.

Theorem 1 if $A \ge B \ge 0$, $A^2 = B^2$, then for any positive integer k, there is $A^k = B^k$.

Proof. Assuming that the rank of A is r and r is smaller than the order of the matrix, there is a reversible matrix Q such that $A = Q^t \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q$, we can know that $A \ge B \ge 0$, $B = Q^t \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} Q$,and $I_r \ge B_1 \ge 0$.Positive definite matrix $QQ^t = \begin{pmatrix} S_1 & S_2 \\ S_2{}^t & S_3 \end{pmatrix}$. $A^2 = Q^t \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} Q$, $B^2 = Q^t \begin{pmatrix} B_1 S_1 B_1 & 0 \\ 0 & 0 \end{pmatrix} Q$, since $A^2 = B^2$,then $S_1 = B_1 S_1 B_1$. S_1 as the main submatrix of the positive definite matrix, so S_1 is a positive matrix, it can be obtained from $S_1 = B_1 S_1 B_1$ that B_1 is a full rank matrix, so B_1 is a positive array, let B_1 be a diagonal array, Then by calculating, the main diagonal elements of B_1 are equal to 1,then $B_1 = I_r$,so $A = B$,and it fit $A^k = B^k$ for any positive integer k .If A is a positive fixed array, and $A^2 = B^2$, so $A = B$,so we have $A^k = B^k$.

Theorem2. If A and B are n-order square matrices on a complex domain, $AA^* \geq BB^*$, then $R(B) \subseteq R(A)$,and $\delta_{\max}(A^+B) \leq 1$.

Proof. Suppose that $A = U\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}V$, Δ is diagonal array, the diagonal element is the square root of the positive eigenvalue of AA^*, $B = U\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}V$,because $AA^* \geq BB^*$,we calculate $B_3 = 0$, $B_4 = 0$,then $B = U\begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}V$,so $B = AK$,for $AA^* \geq BB^*$, $B_1B_1^* + B_2B_2^* \leq \Delta^2$,then $\Delta^{-1}(B_1B_1^* + B_2B_2^*)\Delta^{-1} \leq I_{r(A)}$,we get $\delta_{\max}(A^+B) \leq 1$, $\delta_{\max}(A)$ Indicates the maximum singular value of A. On the contrary, if $R(B) \subseteq R(A)$, and $\delta_{\max}(A^+B) \leq 1$,then $AA^* \geq BB^*$ .If A, B are *Hermite* semi-positive matrix, $R(B) \subseteq R(A)$, $R(B^*) \subseteq R(A^*)$ are known by $A^2 \geq B^2$,and from $\delta_{\max}(A^+B) \leq 1$,we can infer $\lambda_{\max}(A^+B) \leq 1$.So we can get the following Corollarys.

Corollary 1 (*Löwner*) If $0 \leq A^2 \leq B^2$, then $0 \leq A \leq B$.

Corollary 2 (baksalary , Hauke) If $0 \leq A^2 \leq B^2$, then $A \prec B$

Corollary 3 If A, B is a semi-positive matrix, $AA^* \geq BB^*$, then $R(B) \subseteq R(A)$,and $\delta_{\max}(A^+B) \leq 1$.

Theorem 3. If A, B is a semi-positive matrix, $n > 1$ And n is a positive integer, then $A^n \leq B^n, AB = BA \Rightarrow A \leq B$.

Proof. Because A, B is a semi-positive array, and A, B can be exchanged, So there is an orthogonal matrix Q for

$A = Q^t\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}Q$, $B = Q^t\begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix}Q$, and $A^n \leq B^n$ .Then by the nature of the increase function, can know $\lambda_i \leq u_i (i = 1, 2, ..., n)$,so it is known by the nature of the semi-positive matrix that $A \leq B$.

Remarks: When $n = 3$, $A^3 \leq B^3$, $AB = BA$, we can deduce $A \leq B$ .In Proposition 3, the condition that A and B are semi-positive matrices cannot be omitted. Otherwise, there are the following counterexamples. Let $A = -I_2, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we can verify $AB = BA, A^3 = -I_2, B^3 = 0$ .So $A^3 \leq B^3$,but $A \leq B$ is obviously not true.

Proposition 4.If A, B is a semi-positive matrix, and $A^4 \leq B^4$, then $R(A) \subseteq R(B)$, $\delta_{\max}(B^+A) \leq 1$.

Proof. B is a semi-positive matrix, assuming that the rank of B is r, so there exist an orthogonal matrix Q such that $B = Q^t\begin{pmatrix} \Delta_r & 0 \\ 0 & 0 \end{pmatrix}Q$, $\Delta_r = diag\{b_1, b_2, ..., b_r\}$, $b_i > 0$, $i = 1, 2, ...r$ .Suppose that $A = Q^t\begin{pmatrix} A_1 & A_2 \\ A_2^t & A_3 \end{pmatrix}Q$ .Let $A^4 = Q^t\begin{pmatrix} C_1 & C_2 \\ C_2^t & C_3 \end{pmatrix}Q$, $C_3 = (A_1A_2 + A_2A_3)^t(A_1A_2 + A_2A_3) + (A_2{}^tA_2 + A_3{}^2)^2$ can be obtained by calculates. $A^4 \leq B^4$, so $-C_3$ is a semi-positive matrix, but C_3 is a semi-positive matrix, then $C_3 = 0$ .This can be obtained $A_3 = 0$, $A_2 = 0$. So $A = Q^t \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} Q$. Let's set B as a positive matrix, A, B can be contracted at the same time as the diagonal matrix, so we can get $R(A) \subseteq R(B)$ .We use Proposition 2 to know when $A^4 \le B^4$ is established, there must have $A^2 \le B^2$, so $\delta_{\max}(B^+A) \le 1$.So $A^4 \le B^4$ can figure out $\delta_{\max}(B^+A) \le 1$.

Corollary3. If A, B is a semi-positive array, and $A^4 \le B^4$,then exist a matrix K ,such that $A = BK$,and $\lambda(KK^t) \subseteq [0,1]$

For the partial order relationship of the n-th power of the semi-positive definite matrix, we have the conclusion of Proposition 3, but Proposition 3 requires two semi-positive definite exchanges. In fact, this condition is not needed. This is the $L\ddot{o}wner - Heniez$ Theorem.

$L\ddot{o}wner - Heniez$ Theorem. if $A \ge B \ge 0$,then $\forall 0 < r < 1$,have $A^r > B^r > 0$.

Proposition 5 can be obtained immediately by the $L\ddot{o}wner - Heniez$ theorem. If A, B is a semi-positive matrix, and $A^n \le B^n$, then $\lambda_{\max}(B^+A) \le 1$.

But this theorem is not true for any $r > 1$.for example.Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $A^2 - B^2 = \begin{pmatrix} 6 & 5 \\ 5 & 4 \end{pmatrix}$, $AB = \begin{pmatrix} 6 & 1 \\ 2 & 2 \end{pmatrix} \ne \begin{pmatrix} 6 & 2 \\ 1 & 2 \end{pmatrix} = BA$ can be obtained by calculation.

So there are the following theorems: If $A \ge B \ge 0$, $AB = BA$, $f(x)$ is a monotonically increasing function on $(0, +\infty)$,then $f(A) \ge f(B)$.

Proof. Because A, B is a semi-positive array, and A, B can be exchanged, So there is an orthogonal matrix Q for

$A = Q^t \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} Q$, $B = Q^t \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix} Q$, and $\lambda_i \ge u_i$, $\forall i \in [1, n]$.

$f(A) = Q^t \begin{pmatrix} f(\lambda_1) & & & \\ & f(\lambda_2) & & \\ & & \ddots & \\ & & & f(\lambda_n) \end{pmatrix} Q$, $f(B) = Q^t \begin{pmatrix} f(u_1) & & & \\ & f(u_2) & & \\ & & \ddots & \\ & & & f(u_n) \end{pmatrix} Q$, $f(x)$ is a monotonically increasing function on $(0, +\infty)$,so we have $f(\lambda_i) \ge f(u_i)$, $\forall i \in [1, n]$.so $f(A) \ge f(B)$.

From this theorem, the following Corollary can be obtained.

Corollary 4. If $A > B > 0$, then $A^{-1} + B^{-1} > 4(A + B)^{-1}$.

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