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On the Pair of Diophantine Equations $y-x=u^2$, $y^3-x^3=v^2$

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Abstract: *The system of double equations represented by* $y-x=u^2$, $y^3-x^3=v^2$ *is considered for finding their integral solutions for u=1 to 5. A few interesting properties among the solutions are presented.*

Keywords: *System of double equations, integer solutions.*

1. Introduction

A Quartic model of an elliptic curve representing the system of Diophantine equations $x^2-6y^2=-5$, $x=az^2-b$ has been analysed by Mignotte and Petho [1] for $a=2$ *and* $b=1$ by using the Siegal-Baker method. An elementary proof of the above system of equations for $a=2$ *and* $b=1$ has been discussed in [2] by Cohn. In [3], Le proposed an effective method for solving the system of equations $x^2-Dy^2=1-D$, $x=2z^2-1$ where $D-1$ is the power of an odd prime. The pair of pell equations of the form $x^2-ay^2=1$, $y^2-bz^2=1$ where a and b are distinct non-square positive integers has been studied in [4-6]. In [7], it has been proved that the system of equations $x^2-(4m^2-1)y^2=1$, $y^2-bz^2=1$ for positive integers m and b has atmost one positive integer solution. In [8], it has been shown that the system of pell equations $y^2-5x^2=4$, $z^2-442x^2=441$ has no positive integer solutions. In this context, one may refer [9-15]. The above results motivated us to search for the integer solutions for some other choices of special double Diophantine equations. This communication concerns with yet another interesting system of double Diophantine equations namely $y-x=u^2$, $y^3-x^3=v^2$ for its infinitely many non-zero distinct integer solutions.

2. Method of Analysis

The system of double equations to be solved is

$$y-x=u^2 \tag{1}$$

$$y^3-x^3=v^2 \tag{2}$$

Eliminating y between (1) and (2), the resulting equation is

$$4v^2=3u^2(2x+u^2)^2+u^6 \tag{3}$$

Let $Y=2v$, $X=2x+u^2$, $D=3u^2$, $\sigma=u^3$ (4)

Then (3) becomes,

$$Y^2=DX^2+\sigma^2 \tag{5}$$

whose initial solution is (X_0,Y_0).

To find the other solutions of (5), consider the pellian

$$Y^2 = DX^2 + 1 \quad (6)$$

The general solution $\left(\tilde{X}_n, \tilde{Y}_n\right)$ of (6) is given by

$$\tilde{X}_n = \frac{1}{2\sqrt{D}} g_n \, , \tilde{Y}_n = \frac{1}{2} f_n$$

where $f_n = \left(\tilde{Y}_0 + \sqrt{D}\tilde{X}_0\right)^{n+1} + \left(\tilde{Y}_0 - \sqrt{D}\tilde{X}_0\right)^{n+1}$

$$g_n = \left(\tilde{Y}_0 + \sqrt{D}\tilde{X}_0\right)^{n+1} - \left(\tilde{Y}_0 - \sqrt{D}\tilde{X}_0\right)^{n+1}$$

in which $\left(\tilde{X}_0, \tilde{Y}_0\right)$ is the initial solution of (6).

Applying the lemma of Brahmagupta between the solutions $\left(X_0, Y_0\right)$ and $\left(\tilde{X}_n, \tilde{Y}_n\right)$, we have

$$X_{n+1} = \frac{X_0}{2} f_n + \frac{Y_0}{2\sqrt{D}} g_n$$

$$Y_{n+1} = \frac{Y_0}{2} f_n + \frac{X_0}{2} \sqrt{D} g_n$$

In view of (4) and (1), the general values for x and y satisfying (1) and (2) are given by

$$\left.\begin{aligned} x_{n+1} &= \frac{1}{2}\left(X_{n+1} - u^2\right) \\ y_{n+1} &= \frac{1}{2}\left(X_{n+1} + u^2\right) \end{aligned}\right\} \quad (7)$$

To analyse the nature of solutions, one has to go for particular values for u. A few illustrations are given below:

2.1 Illustration: 1

Let $u = 1$ (8)

Then, using (8) in (5), the corresponding equation under consideration is

$$Y^2 = 3X^2 + 1 \quad (9)$$

After performing a few calculations, the corresponding solution to (1), (2) are given by

$$x_{n+1} = \frac{1}{2}\left(X_{n+1} - 1\right)$$

$$y_{n+1} = \frac{1}{2}\left(X_{n+1} + 1\right), \qquad n = -1, 1, 3, 5, ...$$

where $X_{n+1} = \frac{1}{2} f_n + \frac{1}{\sqrt{3}} g_n$

in which $f_n = \left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}$

$$g_n = \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}$$

A few numerical examples are presented in Table:1 below

Table 1. *Numerical Examples*

n	x_{n+1}	y_{n+1}
-1	0	1
1	7	8
3	104	105
5	1455	1456

From the above table, we have the following observations:

1. The values of x_{n+1} *and* y_{n+1} are alternatively even and odd.
2. $26y_{2n+6}-362x_{2n+4}-192$ is a perfect square
3. $12y_{2n+4}-156x_{2n+2}-76$ is a Nasty number
4. $2y_{3n+5}-26x_{3n+3}+6y_{n+3}-78x_{n+1}-56$ is a cubical integer
5. $(2y_{4n+6}-26x_{4n+4}-16)+4(2y_{n+3}-26x_{n+1}-14)^2$ is a Biquadratic integer
6. $x_{n+5}=14y_{n+3}-x_{n+1}-8$
7. Each of the following expressions in Table: 2 represent Hyperbolas:

Table 2. *(Hyperbolas)*

S.no	Hyperbola	(P,Q)
1	$3P^2-Q^2=12$	$(2y_{n+3}-26x_{n+1}-14,\ 45x_{n+1}-3y_{n+3}+24)$
2	$3P^2-Q^2=12$	$(26y_{n+5}-362x_{n+3}-194,\ 627x_{n+3}-45y_{n+5}+336)$
3	$12P^2-Q^2=2352$	$(x_{n+5}-181y_{n+1}+91,\ 627y_{n+1}-3x_{n+5}-315)$

8. Each of the following expressions in Table: 3 represent Parabolas:

Table 3. *(Parabolas)*

S.no	Parabola	(P,Q)
1	$3P-Q^2=12$	$(2y_{2n+4}-26x_{2n+2}-12,\ 45x_{n+1}-3y_{n+3}+24)$
2	$3P-Q^2=12$	$(26y_{2n+6}-362x_{2n+4}-192,\ 627x_{n+3}-45y_{n+5}+336)$
3	$84P-Q^2=2352$	$(x_{2n+6}-181y_{2n+2}+105,\ 627y_{n+1}-3x_{n+5}-315)$

2.2 Illustration: 2

Let $u=2$ (10)

Then, using (10) in (5), the corresponding equation under consideration is

$$Y^2=12X^2+64 \quad (11)$$

Following the procedure as presented above, the corresponding solution to (1), (2) are given by

$$x_{n+1}=\frac{1}{2}(X_{n+1}-4)$$

$$y_{n+1}=\frac{1}{2}(X_{n+1}+4),\qquad n=-1,0,1,2,....$$

where $X_{n+1}=2f_n+\frac{4}{\sqrt{3}}g_n$

in which $f_n=(7+2\sqrt{12})^{n+1}+(7-2\sqrt{12})^{n+1}$

$$g_n=(7+2\sqrt{12})^{n+1}-(7-2\sqrt{12})^{n+1}$$

A few numerical examples are presented in Table: 4 below

Table 4. *Numerical examples*

n	x_{n+1}	y_{n+1}
-1	0	4
0	28	32
1	416	420
2	5820	5824

From the above table, we have the following observations:

1. The values of x_{n+1} *and* y_{n+1} are always even
2. $\frac{1}{2}[x_{2n+3}-13x_{2n+2}-20]$ is a perfect square
3. $3x_{2n+3}-39x_{2n+2}-60$ is a Nasty number
4. $\frac{1}{2}[x_{3n+4}-13x_{3n+3}+3x_{n+2}-39x_{n+1}-96]$is a cubical integer
5. $\frac{1}{2}(x_{4n+5}-13x_{4n+4}-28)+(x_{n+2}-13x_{n+1}-24)^2$ is a Biquadratic integer
6. $x_{n+3}=14x_{n+2}-x_{n+1}+24$
7. $y_{n+2}=x_{n+2}+4$
8. $y_{n+3}=14x_{n+2}-x_{n+1}+28$
9. Each of the following expressions in Table: 5 represent hyperbolas:

Table 5. *(Hyperbolas)*

S.no	Hyperbola	(P,Q)
1	$12P^2-Q^2=192$	$(x_{n+2}-13x_{n+1}-24,\ 45x_{n+1}-3x_{n+2}+84)$
2	$12P^2-Q^2=37632$	$(x_{n+3}-181x_{n+1}-360,\ 627x_{n+1}-3x_{n+3}+1248)$
3	$12P^2-Q^2=192$	$(13y_{n+3}-181x_{n+2}-388,\ 627x_{n+2}-45y_{n+3}+1344)$

10. Each of the following expressions in Table: 6 represent parabolas:

Table 6. *(Parabolas)*

S.no	Parabola	(P,Q)
1	$24P-Q^2=192$	$(x_{2n+3}-13x_{2n+2}-20,\ 45x_{n+1}-3x_{n+2}+84)$
2	$336P-Q^2=37632$	$(x_{2n+4}-181x_{2n+2}-304,\ 627x_{n+1}-3x_{n+3}+1248)$
3	$24P-Q^2=192$	$(13y_{2n+4}-181x_{2n+3}-384,\ 627x_{n+2}-45y_{n+3}+1344)$

2.3 Illustration: 3

Let $u=3$ (12)

Then, using (12) in (5), the corresponding equation under consideration is

$$Y^2=27X^2+729 \tag{13}$$

After some algebra, the corresponding solution to (1), (2) are found to be

$$x_{n+1}=\frac{1}{2}(X_{n+1}-9)$$

$$y_{n+1}=\frac{1}{2}(X_{n+1}+9), \qquad n=-1,1,3,5,....$$

where $X_{n+1}=\frac{9}{2}f_n+\sqrt{27}g_n$

in which $f_n=(26+15\sqrt{3})^{n+1}+(26-15\sqrt{3})^{n+1}$

$$g_n=(26+15\sqrt{3})^{n+1}-(26-15\sqrt{3})^{n+1}$$

A few numerical examples are presented in Table:7 below

Table 7. *Numerical examples*

n	x_{n+1}	y_{n+1}
-1	0	9
1	13095	13104
3	35394840	35394849

From the above table, we have the following observations:

1. The values of x_{n+1} *and* y_{n+1} are alternatively even and odd
2. $\frac{1}{1755}\left[2x_{2n+4}-5042x_{2n+2}-19170\right]$ is a perfect square
3. $\frac{4}{585}\left[x_{2n+4}-2521x_{2n+2}-9585\right]$is a Nasty number
4. $\frac{1}{1755}\left[2x_{3n+5}-5042x_{3n+3}+6x_{n+3}-15126x_{n+1}-90720\right]$is a cubical integer
5. $x_{n+5}=2702x_{n+3}-x_{n+1}+12150$
6. Each of the following expressions in Table: 8 represent hyperbolas

Table 8. *(Hyperbolas)*

S.no	Hyperbola	(P,Q)
1	$P^2-3Q^2=12320100$	$\begin{pmatrix} 2x_{n+3}-5042x_{n+1}-22680, \\ 2911x_{n+1}-x_{n+3}+13095 \end{pmatrix}$
2	$4P^2-3Q^2=16(2371005)^2$	$\begin{pmatrix} y_{n+5}-6811741x_{n+1}-30652839, \\ 7865521x_{n+1}-y_{n+5}+35394849 \end{pmatrix}$

7. Each of the following expressions in Table: 9 represent parabolas

Table 9. *(Parabolas)*

S.no	Parabola	(P,Q)
1	$585P-Q^2=4106700$	$\begin{pmatrix} 2x_{2n+4}-5042x_{2n+2}-19170, \\ 2911x_{n+1}-x_{n+3}+13095 \end{pmatrix}$
2	$3161340P-Q^2$ $=108(526890)^2$	$\begin{pmatrix} y_{2n+6}-6811741x_{2n+2}-25910829, \\ 7865521x_{n+1}-y_{n+5}+35394849 \end{pmatrix}$

2.4 Illustration: 4

Let $u=4$ (14)

Then, using (14) in (5), the corresponding equation under consideration is

$$Y^2=48X^2+4096 \quad (15)$$

After performing a few calculations, the corresponding solution to (1), (2) are given by

$$x_{n+1}=\frac{1}{2}(X_{n+1}-16)$$

$$y_{n+1}=\frac{1}{2}(X_{n+1}+16), \qquad n=-1,0,1,2,....$$

where $X_{n+1}=6f_n+\sqrt{48}g_n$

in which $f_n=\left(7+4\sqrt{3}\right)^{n+1}+\left(7-4\sqrt{3}\right)^{n+1}$

$$g_n=\left(7+4\sqrt{3}\right)^{n+1}-\left(7-4\sqrt{3}\right)^{n+1}$$

A few numerical examples are presented in Table: 10 below

Table 10. *Numerical examples*

n	x_{n+1}	y_{n+1}
-1	-2	14
0	82	98
1	1246	1262
2	17458	17474

From the above table, we have the following observations:

1. The values of x_{n+1} *and* y_{n+1} are always even
2. $\frac{1}{84}\left[x_{2n+4}-181x_{2n+2}-1272\right]$ is a perfect square
3. $\frac{1}{14}\left[x_{2n+4}-181x_{2n+2}-1272\right]$is a Nasty number
4. $\frac{1}{84}\left[x_{3n+5}-181x_{3n+3}+3x_{n+3}-543x_{n+1}-5760\right]$is a cubical integer
5. $14x_{n+2}=x_{n+3}+x_{n+1}-96$
6. $y_{n+3}=x_{n+3}+16$
7. $14y_{n+2}=x_{n+3}+x_{n+1}+128$
8. Each of the following expressions in Table: 11 represent hyperbolas:

Table 11. *(Hyperbolas)*

S.no	Hyperbola	(P,Q)
1	$4P^2-3Q^2=112896$	$\left(x_{n+3}-181x_{n+1}-1440,\ 209x_{n+1}-x_{n+3}+1664\right)$
2	$4P^2-3Q^2=576$	$\left(13y_{n+3}-181x_{n+2}-1552,\ 209x_{n+2}-15y_{n+3}+1792\right)$
3	$4P^2-3Q^2=576$	$\left(y_{n+2}-13x_{n+1}-112, 15x_{n+1}-y_{n+2}+128\right)$

9. Each of the following expressions in Table: 12 represent parabolas:

Table 12. *(Parabolas)*

S.no	Parabola	(P,Q)
1	$112P-Q^2=37632$	$\left(x_{2n+4}-181x_{2n+2}-1272,\ 209x_{n+1}-x_{n+3}+1664\right)$
2	$8P-Q^2=192$	$\left(13y_{2n+4}-181x_{2n+3}-1540,\ 209x_{n+2}-15y_{n+3}+1792\right)$
3	$8P-Q^2=192$	$\left(y_{2n+3}-13x_{2n+2}-100, 15x_{n+1}-y_{n+2}+128\right)$

2.5 Illustration: 5

Let $u=5$ (16)

Then, using (16) in (5), the corresponding equation under consideration is

$$Y^2=75X^2+15625 \qquad (17)$$

Following the procedure as presented above, the corresponding solution to (1), (2) are given by

$$x_{n+1}=\frac{1}{2}\left(X_{n+1}-25\right)$$

$$y_{n+1}=\frac{1}{2}\left(X_{n+1}+25\right), \qquad n=-1,1,3,5,....$$

where $X_{n+1}=\frac{25}{2}f_n+\frac{25}{\sqrt{3}}g_n$

in which $f_n=\left(26+15\sqrt{3}\right)^{n+1}+\left(26-15\sqrt{3}\right)^{n+1}$

$g_n=\left(26+15\sqrt{3}\right)^{n+1}-\left(26-15\sqrt{3}\right)^{n+1}$

A few numerical examples are presented in Table: 13 below

Table 13. *Numerical examples*

n	x_{n+1}	y_{n+1}
-1	0	25
1	36375	36400
3	98319000	98319025
5	265657935375	265657935400

From the above table, we have the following observations:

1. The values of x_{n+1} *and* y_{n+1} are alternatively even and odd
2. $\frac{1}{14625}\left[6x_{2n+4}-15126y_{2n+2}+218400\right]$ is a perfect square
3. $\frac{2}{4875}\left[6x_{2n+4}-15126y_{2n+2}+218400\right]$is a Nasty number
4. $\frac{1}{14625}\left[6x_{3n+5}-15126y_{3n+3}+18x_{n+3}-45378y_{n+1}+756600\right]$is a cubical integer
5. Each of the following expressions in Table: 14 represent hyperbolas

Table 14. *(Hyperbolas)*

S.no	Hyperbola	(P,Q)
1	$P^2-27Q^2=855562500$	$\left(6x_{n+3}-15126y_{n+1}+189150,\ 2911y_{n+1}-x_{n+3}-36400\right)$
2	$4P^2-3Q^2=16(6586125)^2$	$\left(y_{n+5}-6811741x_{n+1}-85146775,\ 7865521x_{n+1}-y_{n+5}+98319025\right)$

6. Each of the following expressions in Table: 15 represent parabolas

Table 15. *(Parabolas)*

S.no	Parabola	(P,Q)
1	$1625P-3Q^2=95062500$	$\left(6x_{2n+4}-15126y_{2n+2}+218400,\ 2911y_{n+1}-x_{n+3}-36400\right)$
2	$8781500P-Q^2$ $=12(4390750)^2$	$\left(y_{2n+6}-6811741x_{2n+2}-71974525,\ 7865521x_{n+1}-y_{n+5}+98319025\right)$

REFERENCES

[1] Mignotte M., Petho A., On the system of Diophantine equations $x^2-6y^2=-5$, $x=az^2-b$, Mathematica Scandinavica, 76(1), 50-60, (1995).

[2] Cohn JHE., The Diophantine system $x^2-6y^2=-5$, $x=2z^2-1$, Mathematica Scandinavica, 82(2),161-164, (1998).

[3] Le MH., On the Diophantine system $x^2-Dy^2=1-D$, $x=2z^2-1$, Mathematica Scandinavica, 95(2),171-180, (2004).

[4] Anglin W.S., Simultaneous pell equations, Maths. Comp. 65, 355-359, (1996).

[5] Baker A., Davenport H., The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, Quart. Math. Oxford, 20(2), 129-137, (1969).

[6] Walsh P.G., On integer solutions to $x^2 - dy^2 = 1$ and $z^2 - 2dy^2 = 1$, Acta Arith. 82, 69-76, (1997).

[7] Mihai C., Pairs of pell equations having atmost one common solution in positive integers, An.St.Univ.Ovidius Constanta, 15(1), 55-66, (2007).

[8] Fadwa S. Abu Muriefah and Amal Al Rashed, The simultaneous Diophantine equations $y^2 - 5x^2 = 4$ and $z^2 - 442x^2 = 441$, The Arabian Journal for Science and engineering, 31(2A), 207-211, (2006).

[9] Gopalan M.A., Vidhyalakshmi S., and Lakshmi K., On the system of double equations $4x^2 - y^2 = z^2$, $x^2 + 2y^2 = w^2$, Scholars Journal of Engineering and Technology (SJET), 2(2A), 103-104, (2014).

[10] Gopalan M. A., Vidhyalakshmi S., and Janani R., On the system of double Diophantine equations $a_0 + a_1 = q^2$, $a_0 a_1 \pm 2(a_0 + a_1) = p^2 - 4$, Transactions on Mathematics™, 2(1), 22-26, (2016).

[11] Gopalan M.A., Vidhyalakshmi S., and Nivetha A., On the system of double Diophantine equations $a_0 + a_1 = q^2$, $a_0 a_1 \pm 6(a_0 + a_1) = p^2 - 36$, Transactions on Mathematics™, 2(1), 41-45, (2016).

[12] Gopalan M. A., Vidhyalakshmi S., and Bhuvaneswari E., On the system of double Diophantine equations $a_0 + a_1 = q^2$, $a_0 a_1 \pm 4(a_0 + a_1) = p^2 - 16$, Jamal Academic Research Journal, Special Issue, 279-282, (2016).

[13] Meena K., Vidhyalakshmi S., and Priyadharsini C., On the system of double Diophantine equations $a_0 + a_1 = q^2$, $a_0 a_1 \pm 5(a_0 + a_1) = p^2 - 25$, Open Journal of Applied & Theoretical Mathematics (OJATM), 2(1), 08-12, (2016).

[14] Gopalan M.A., Vidhyalakshmi S., and Rukmani A., On the system of double Diophantine equations $a_0 - a_1 = q^2$, $a_0 a_1 \pm (a_0 - a_1) = p^2 + 1$, Transactions on Mathematics™, 2(3), 28-32, (2016).

[15] Devibala S., Vidhyalakshmi S., Dhanalakshmi G., On the system of double equations $N_1 - N_2 = 4k + 2\ (k \succ 0)$, $N_1 N_2 = (2k+1)\alpha^2$, International Journal of Engineering and Applied Sciences (IJEAS), 4(6), 44-45, (2017).

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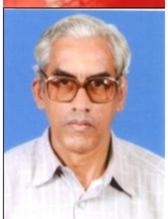

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