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On Oscillatory Behavior of Solutions of Third-Order Nonlinear Differential Equations

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Abstract: In this paper we investigate the behavior of the bounded solutions of a class of third order nonlinear differential equations $\left(r(x)(r(x)y^1(x))^1\right)^1 + q(x)y^1(x) + p(x)y^\lambda(x) = 0$ has been studied with $p(x) < 0, q(x) \geq 0, q^1(x) \geq 0$forx $\in (a,\infty)$.Also the necessary and sufficient condition for any of its solutions to be oscillatory was considered.

Keywords: *Oscillatory solution, nonoscillatory solution, oscillation, nonlinear differential equations.*

1. Introduction

The aim of this paper is study the properties of oscillatory solutions of nonlinear differential equations of third order

$$\left(r(x)\left(r(x)y^1(x)\right)^1\right)^1 + q(x)y^1(x) + p(x)y^\lambda(x) = 0, \lambda > 0 \tag{1.1}$$

Where $p(x), q(x), r(x)$ are continuous functions, with $r(x) > 0$ on the interval $I \in (a,\infty), -\infty < a$ and λ is positive and is a ratio of two odd integers. The equation (1.1) can be termed as such super linear, linear and sublinear according as $\lambda > 1, \lambda = 1$ and $\lambda < 1$respectively. A nontrivial solution of **(1.1)** is said to be oscillatory if it has zeros for arbitrarily large values of the independent variable, otherwise it is said to be nonoscillatory. In the linear case with $\lambda = 1$and $r(x) = 1$the properties of oscillatory solutions are discussed in Hanan [01], Lazer [02] and Swanson [03]. The associated nonlinear equations with $r(x) = 1$is the subject discussed by Waltman [04], Heidal [05], Gragus and Venko [06], N.Parhi and S.Parhi [07] have discussed the equation **(1.1)** with $q(x) = 0$in the form

$$y^{111}(x) + p(x)y(x) = 0 \tag{1.2}$$

Also the behavior of solutions of the equations $y^{111}(x) + p(x)y^{11}(x) + q(x)y^1(x) + r(x)y^\lambda(x) = 0$ (1.3) was considered for oscillation, nonoscillation and asymptotic behavior by L.Erbe[08] via a second order equation.

In the discussion of the oscillatory solutions of (1.1)made in the section no recourse has been taken to second order equations as well as no change of variables incorporated. The results obtained generalize many theorems mentioned in the above references. To the best of our knowledge nothing is known regarding the behavior of the bounded solutions of a class of third order nonlinear differential equation.

2. Main Results

In this section we study the equation

$$\left(r(x)(r(x)y^1(x))^1\right)^1 + q(x)y^1(x) + p(x)y^\lambda(x) = 0 \tag{2.1}$$

with$p(x) < 0, q(x) \geq 0, q^1(x) \geq 0$ for $x \in (a, \infty)$. We prove a theorem on bounded solutions to be oscillatory and a theorem on necessary and sufficient conditions that a solution of equation (2.1.1) to be oscillatory. We first prove a lemma that is needed for the purpose.

2.1 LEMMA

Let

$$y(x) \in C^3([x_0, \infty)) > 0 \tag{2.3}$$

$$r(x) \in C([x_0, \infty)),\ r(x) > 0 \text{ and bounded for } x \in [x_0, \infty) \tag{2.4}$$

Then

$$y(x) > 0, (ry^1(x)) < 0 \text{ and } (r(ry^1)^1)^1(x) > 0 \tag{2.5}$$

Cannothold for all $x \geq x_0$.

PROOF:-

Suppose for contradiction that (2.5) holds. Consider $(r(ry^1)^1)^1(x) > 0$ for $x \geq x_0$.

This implies $r(x)(ry^1)^1(x)$ increase for all $x \geq x_0$.

Since $r(x)$ is positive and bounded we have, $(ry^1)^1(x)$ increases for$x \in [x_0, \infty)$. (2.6)

Suppose

$$(ry^1)^1(x) > 0, \tag{2.7}$$

which implies

$$r(x)y^1(x) \text{ increases for } x \in [x_0, \infty) \tag{2.8}$$

Let $(ry^1)^1(x) > k$ for some $k > 0$.

Integrating between x_0 and x ,we get

$$r(x)y^1(x) > k(x - x_0) + r(x_0)y^1(x_0) \tag{2.9}$$

The inequality holds in case $(ry)^1(x) > 0$, which is a contradiction to our supposition that (2.5) holds.

Now suppose

$$(ry^1)^1(x) < k \text{ for some k} < 0,\ x \in [x_0, \infty) \tag{2.10}$$

Integrating between x_0 and x ,we obtain

$$r(x)y^1(x) < k(x - x_0) + r(x_0)y^1(x_0) \tag{2.11}$$

< 0, for sufficiently large x.

Since $r(x)$ is positive and bounded for$x \in [x_0, \infty)$ the inequality (2.11) implies

$y^1(x) < 0$ for $x \in [x_0, \infty)$ and $y(x)$ decreases.

Suppose

$$y(t) > 0 \text{ for } x \in [x_0, \infty) \tag{2.12}$$

Let $y^1(x) < m$ for some $m < 0$ for $x \in [x_0, \infty)$.

Multiplying with $y(x)$ both sides and integrating between x_0 and x, we obtain

$$\int_{x_0}^{x} y(s)y^1(s)ds < m\int_{x_0}^{x} y(s)ds$$

$$\frac{y^2(x)}{2} < m\int_{x_0}^{x} y(s)ds + \frac{y^2(x_0)}{2} \tag{2.13}$$

The left side of (2.13) is positive. While the right side is negative for sufficiently large x. So $y(x) > 0$ cannot hold for all $x \in [x_0, \infty)$. This is a contradiction to our supposition, that (2.5) holds.

Suppose now that

$$(r(ry^1)^1)^1(x) = 0 \text{ for } x \geq x_0 \tag{2.14}$$

This implies

$$r(x)(ry^1)^1(x) = k_1, \text{ a constant.} \tag{2.15}$$

Since $r(x) > 0$ and bounded for $x > x_0$ (2.14) implies

$$(ry^1)^1(x) = k_2, \text{ a constant.} \tag{2.16}$$

Incase $(ry^1)^1(x) = k_2 > 0,$ an integration produces

$$r(x)y^1(x) = k_2(x - x_0) + r(x_0)y^1(x_0) \tag{2.17}$$

$$> 0, \text{ for sufficiently large } x.$$

This is a contradiction to our supposition.

Incase $(ry^1)^1(x) = k_2 < 0,$ an integration produces

$$r(x)y^1(x) = k_2(x - x_0) + r(x_0)y^1(x_0) \tag{2.18}$$

$< 0,$ for sufficiently large x.

This implies $y^1(x) < 0$ since $r(x) > 0$ and bounded.

Let $y^1(x) < k$, for some $k < 0$.

An integration between x_0 to x produces

$$y(x) < k(x - x_0) + y(x_0) \tag{2.19}$$

< 0

Which is a contradiction to our supposition that (2.5) holds. Hence the Lemma.

The main objective of this paper is to prove the following:-

2.2 THEOREM

Let $p(x) < 0,\ q(x) \geq 0,\ q^1(x) \geq 0,\ r(x) > 0$ for $x \in (a, \infty)$ (2.21)

$r(x), q(x)$ be bounded on (a, ∞) (2.22)

and

$$\int_{x_0}^{\infty} p(s)ds = -\infty,\ x_0 > a \tag{2.23}$$

Then every bounded solution of (2.18) on $x\in[x_0,\infty)$ is oscillatory on this interval.

PROOF:-

Assume without lossof generality that $y(x)>0$ be bounded on $[x_0,\infty)$, $x_0>a$.

Three cases arise that depend upon the sign of $r(x)y^1(x)$.

CASE(i):

$r(x)y^1(x)>0\, x\geq X\geq x_0$

Integrating (2.1) between x_0 and x , we get

$$r(x)(ry^1)^1(x)+q(x)y(x)+\int_{x_0}^{x}p(s)y^{\lambda-1}(s)-q^1(s)y(s)ds=k \tag{2.24}$$

Where

$k=r(x_0)(ry^1)^1(x_0)+q(x_0)y(x_0)$, a constant.

Using (2.21), (2.22) and (2.23) and that $y(x)$ bounded,implies $(ry^1)^1(x)\to\infty$ as $t\to\infty$ (2.25)
Therefore $y(x)$ cannot be bounded on $x\in[x_0,\infty)$, which is a contradiction.

CASE(ii):-

$r(x)y^1(x)\leq 0$ for $x\geq X\geq x_0$.

Writing equation (2.1) as

$\left(r(x)(r(x)y^1(x))^1\right)^1=-q(x)y^1(x)-p(x)y^{\lambda}(x)$ (2.25) $>-q(x)r(x)y^1(x)-p(x)y^{\lambda}(x)$

>0 for $x\geq X$, using (2.21).

But by Lemma 2.1 this is impossible.

CASE(iii):-

$(ry^1)(x)$ has infinitely many null points at which it changes its sign.

Let $y(x)>k>0$, then from (2.24) we obtain,

$$(ry^1)^1(x)>0 \text{ for } x\geq X\geq x_0 \tag{2.26}$$

and this implies that $(ry^1)(x)>0$ increases for $x\geq X$.

This is a contradiction with the case $(ry^1)(x)$ is oscillatory.

Therefore $\liminf_{x\to\infty} y(x)=0.$

Since $r(x)$ is positive and bounded.

$$\text{Suppose}, \liminf_{x\to\infty} y(x)=0, \tag{2.27}$$

We have the following subcases.

Subcase (i):-

Let

$$\int_{x_0}^{\infty}[p(s)y^{\lambda-1}(s)-q^1(s)]y(s)ds=\infty \tag{2.28}$$

Then [2.25] implies that

$(ry^1)^1(x) > 0$ for $x \geq X \geq x_0$ T.

This is contradiction to the oscillatoricity of $(ry^1)(x)$.

Subcase(ii):-

Let

$$0 \leq -\int_{x_0}^{\infty} [p(s)y^{\lambda-1}(s) - q^1(s)]y(s)ds < \infty \tag{2.29}$$

Let $\langle x_i \rangle_{i=1}^{\infty}$, $x_i \to \infty$ for $i \to \infty$, be a sequence of points at which $(ry^1(x_i)) = 0$ and $(ry^1)^1(x_i) > 0$.

It follows from (2.22) that $(ry^1)^1(x_i)$ is bounded on $[x_0, \infty)$.

Multiplying equation (2.1) by the solution $y(x)$ and integrating from x_i to x ,

we obtain

$$\int_{x_i}^{x} (r(ry^1)^1)^1(s)y(s)ds + \int_{x_i}^{x} q(s)y^1(s)y(s)ds + \int_{x_i}^{x} p(s)y^{\lambda+1}(s)ds = 0 \tag{2.30}$$

Integrating by parts yields

$$r(x)y(x)(ry^1)^1(x) - \frac{1}{2}(ry^1)^2(x) + \frac{1}{2}q(x)(y^1)^2(x) + \int_{x_i}^{x} [p(s)y^{\lambda-1}(s) - \frac{1}{2}q^1(s)]y^2(s)ds$$

$$= r(x_i)y(x_i)(ry^1)^1(x_i) - \frac{1}{2}(ry^1)^2(x_i) + \frac{1}{2}q(x_i)y^2(x_i),\ i = 1,2,3,\ldots\ldots\ldots \tag{2.31}$$

We write the above equation in the form

$$r(x)y(x)(ry^1)^1(x) - \frac{1}{2}(ry^1)^2(x) + \frac{1}{2}q(x)(y^1)^2(x) + \int_{x_i}^{x} [p(s)y^{\lambda-1}(s) - \frac{1}{2}q^1(s)]y^2(s)ds = k_i \tag{2.32}$$

Where $k_i = r(x_i)y(x_i)(ry^1)^1(x_i) - \frac{1}{2}(ry^1)^2(x_i) + \frac{1}{2}q(x_i)y^2(x_i),\ i = 1,2,3,\ldots\ldots\ldots$

From this we obtain for $(ry^1)^1(x_i), i = 1,2,3,\ldots\ldots$

$$(ry^1)^1(x_i) = \frac{k_i}{r(x_i)y(x_i)} - \frac{1}{2}\frac{q(x_i)(y^1(x_i))^2}{r(x_i)y(x_i)} - \frac{1}{r(x_i)y(x_i)}\int_{x_i}^{x} [p(s)y^{\lambda-1}(s) - \frac{1}{2}q^1(s)]y^2(s)ds.$$

From this it follows that $x_i \to \infty$, $(ry^1)^1(x_i) \to \infty$.

Which is a contradiction with the boundedness of $\{(ry^1)^1(x_i)\}$. this complete the proof.

2.3 LEMMA

Let

$$p(x) < 0, q(x) \geq 0, q^1(x) \geq 0, r(x) > 0 \tag{2.34}$$

for $x \in (a, \infty)$

$$r(x),\ q(x) \text{ be on bounded } x \in (a, \infty) \tag{2.35}$$

and

$$\int_{x_0}^{\infty} p(s)\,ds = -\infty,\ x_0 > a \tag{2.36}$$

Then for every $y(x)$ of (2.1) with the property $y(x) > 0$ for $x > x_0$, there exists $X \geq x_0$ such that for $x \geq X$ the inequality

$$r(x)y(x)(ry^1)^1(x) - \frac{1}{2}(y^1(x))^2 + q(x)y^2(x) > 0 \text{ holds.} \tag{2.37}$$

PROOF:-

Let $y(x) > 0$ for $x \geq x_0$. Then there are three cases for $(ry^1)(x)$.

CASE(i):-

$(ry^1)(x) > 0$ for $x \geq x_0$.

It follows from (2.32) with $x_i = x_0$ that there exists $X \geq x_0$ such that for all $x \geq X$,the inequality (2.37) holds.

CASE(ii):-

$(ry^1)(x) \leq 0$ for $x \geq x_0$

Writing (2.1) as

$$\left(r(x)(r(x)y^1(x))^1\right)^1 = -q(x)y^1(x) - p(x)y^\lambda(x)$$
$$> 0 \text{ using (2.34)}$$

But using Lemma 2.2 this is not possible.

CASE(iii):-

$(ry^1)(x)$ has on $[x_0, \infty)$ at least two null points at which it changes sign.

At one of the null points we have $(ry^1)^1(x) \geq 0$.

Let $x = X_i$. It follows from (2.32) with $x_i = X$ that

$$k_i = r(X_i)y(X_i)(ry^1)^1(X_i) + \frac{1}{2}q(X_i)y^2(X_i),\ i = 1,2,3,\ldots\ldots \geq 0 \tag{2.38}$$

Such that (2.37) holds for $x > X$. Thus the lemma is proved.

2.4 THEOREM

Let

$$p(x) < 0, q(x) \geq 0, q^1(x) \geq 0, r(x) > 0 \tag{2.40}$$

for $x \in (a, \infty)$

$$r(x),\ q(x) \text{ be on bounded } x \in (a, \infty) \tag{2.41}$$

And $$\int_{x_0}^{\infty} p(s)\,ds = -\infty,\ x_0 > a \tag{2.42}$$

Then a necessary and sufficient condition for the solution $y(x)$ of (2.1) define on $[x_0, \infty), x_0 > a$ to be oscillatory on $[x_0, \infty)$ is that

$$r(x)y(x)(ry^1)^1(x)-\frac{1}{2}(ry^1)^2(x)+\frac{1}{2}q(x)y^2(x)<0 \text{ for } x\geq X\geq x_0 \tag{2.43}$$

PROOF:-

We prove the condition (2.43) is sufficient. Let $y(x)$ be a solution of (2.1) which satisfies the condition (2.43). Let us assume without loss of generality that $y(x)>0$ for $x\geq X$. But then by lemma 2.33 there exists $X_1\geq x_0$ such that (2.42.) holds for $x\geq X_1$ and this contradicts (2.43). Thus must be oscillatory.

We shall now prove the condition (2.43) is necessary. That is we must prove that an oscillatory solution in $[x_0,\infty)$ fulfils the condition (2.43). Let $y(x)$ be an oscillatory solution of (2.1) on $[x_0,\infty)$ and let $x_i, i=1,2,3,\ldots\ldots\ldots\ \ldots\ldots\ldots$ be its null points on $[x_0,\infty)$. Let us denote $G(y(x))$ by

$$G(y(x)):=r(x)y(x)(ry^1)^1(x)-\frac{1}{2}(ry^1)^2(x)+\frac{1}{2}q(x)y^2(x) \tag{2.44}$$

Equation (2.30) can be written as

$$G(y(x)):=$$

$$G(y(x_i))-\int_{x_i}^{x}[p(s)y^{\lambda-1}(s)-\frac{1}{2}q^1(s)]y^2(s)ds \tag{2.45}$$

$$\frac{dG(y(x))}{dx}=-[p(x)y^{\lambda-1}(x)-\frac{1}{2}q^1(x)]y^2(x) \tag{2.46}$$

>0 on $[x,\infty)$ using (2.1)

(2.44) implies that $G(y(x))$ is increasing on $[x_1,\infty)$ and thus

$$G(y(x)):=r(x_i)y(x_i)(ry^1)^1(x_i)-\frac{1}{2}(ry^1)^2(x_i)+\frac{1}{2}q(x)y^2(x_i)<0 \text{ for } i=1,2,3,\ldots\ldots\ldots \tag{2.47}$$

and so (2.42) holds. Hence the proof.

3. Conclusion

In this paper, we considered the third order non-linear homogeneous differential equations. Let $y(x)$ be a solution of (2.1) with the property $y(x_0)=0,\ (ry^1)(x_0)=0,\ (ry^1)^1(x_0)>0$ and let the hypothesis of theorem (2.37)be satisfied. Then $y(x)>0$ for $x>x_0$.

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