International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 5, Issue 3, 2017, PP 9-13
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
DOI: http://dx.doi.org/10.20431/2347-3142.0503002
www.arcjournals.org

Some Results on Fixed Points of Nonlinear Contraction in Metric Space

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Abstract*: In this paper, I have generalized the result of Sayyed and Sayyed [17] in metric spaces by replacing the containment condition and giving the shorter proof than of the authors in the main result*.

Keywords and Phrases: *Common fixed point, Compatible mapping, Property (E.A.), Common property (E.A.), Occasionally weakly compatible maps, Coincidence points.*

1. Introduction

Aamri and Moutawakil [1] introduced the concept of property (E.A.) which was perhaps inspired by the condition of compatibility introduced by Jungck [11] and further Imdad and Ali[10] extended this result . Recently Babu and Alemayehu [7, 8,9] proved common fixed point theorem for occasionally weakly compatible maps satisfying property (E.A.) using an inequality involving quadratic terms. Aliouche[4] proved a common fixed point theorem of Gregus type weakly compatible mappings satisfying generalized contractive conditions.

Abbas [2] established a common fixed point for Lipschitzian mapping satisfying rational contractive conditions. Murty et.al [15] proved fixed points of nonlinear contraction in metric space.

2. Preliminaries

Throughout this paper (X, d) is a metric space which is denoted by X.

Definition 2.1: [Jungck and Rhoades [13]]. Let A and S be selfmaps of a set X. If $Au = Su = \omega$ (say), $\omega \in X$, for some u in X, then u is called a coincidence point of A and S and the set of coincidence points of A and S is denoted by $C(A, S)$, and ω is called a point of coincidence of A and S.

Definition 2.2: Let A, B, S and T be self maps of a set X. If $u \in C(A,S)$ and $v \in C(B,T)$ for some $u, v \in X$ and $Au = Su = Bv = Tv = z$ (say), then z is called a common point of coincidence of the pairs $(A. S)$ and $(B. T)$.

Definition 2.3: The pair (A, S) is said to be

(I) Satisfy property $(E.A.)$ [1] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some t in X.

(II) Copatible [11] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, for some t in X whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$.

(III) Weakly compatible [12], if they commute at their coincidence point.

(IV) Occasionally weakly compatible (owc) [3,5,6] if $ASx = SAx$ for some $x \in C(A,S)$.

Remark 2.4

(I) Every compatible pair is weakly compatible but its converse need not be true [12].

(II) Weak compatibility and property (*E. A.*) are independent of each other [16].

(III) Every weakly compatible pair is occasionally weakly compatible but its converse need not be true [11].

(IV) Occasionally weakly compatible and property (*E.A.*) are independent of each other [8].

Definition 2.5: [14] Let (*X, d*) be a metric space and *A, B, S* and *T* be four selfmaps on X. The pairs (A,S) and (B,T) are said to satisfy common property (*E.A.*) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n$ for some t in X.

Remark 2.6: Let A, B, S and *T* be self maps of a set X. If the pairs (*A, S*) and (*B, T*) have common point of coincidence in *X* then *C*(*A, S*) ≠ ϕ and *C* (*B, T*) ≠ ϕ. But converse is not true.

Example 2.7: Let *X* = [0, ∞) with usual metric and *A, B, S* and *T* self maps on *x* and defined by $Ax = 1 - x^2; sx = 1 - x; Bx = \frac{1}{2} + x^2; Tx = \frac{1+x}{2}$ for all $x \in X$.

It is easy to observe that $C(A,S) = \{0,1\}$ and $C(B,T) = \left\{0, \frac{1}{2}\right\}$ but the pairs (*A, S*) and (*B, T*) not having common point of coincidence.

Remark 2.8: The converse of the remark 2.6 is true provided it satisfies inequality (3.1). This is given as in proposition 3.1 in section III.

Preposition 2.9: [2] Let *A* and *S* be two self maps of a set *X* and the pair *(A, S)* is satisfies occasionally weakly compatible (owc) condition. If the pairs (*A, S*) have unique point of coincidence *Ax* = *Sx* = *z* then *z* is the unique common fixed point of *A* and *S*.

Proof: To be given *Ax* = *Sx* = {*z*} (say) for any $x \in C(A,S)$ (2.1)

Since the pair (*A, S*) satisfies the property owc, therefore

$$Az = ASx = SAx = Sz \text{ implies that } z \in C(A,S)$$

From (2.1), we get $Az = Sz = z$. Hence proposition follows.

In 1996, Tas et al. [17] proved the following.

Theorem 2.10: Let *A, B, S* and *T* be selfmaps of a complete metric space (X,d) such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ and satisfying the inequality.

$$[d(Ax,By)]^2 \le C_1 \max\left\{[d(Sx,Ax)]^2, [d(Ty,By)]^2, [d(Sx,Ty)]^2\right\}$$
$$+C_2 \max\{d(Sx,Ax)d(Sx,By), d(Ty,Ax)d(Ty,By)\}$$
$$+C_3 d(Sx,By)d(Ty,Ax)$$

for all $x, y \in x$, where $C_1 + C_3, C_2, C_3 \ge 0, C_1 + 2C_2 < 1, C_1 + C_3 < 1$. Further, assume that the pairs (A,S) and (B,T) are compatible onX. If one of the mappings A, B, S and *T* is continuous then A, B, S and *T* have a unique common fixed point in *X*.

3. MAIN RESULTS

Proposition 3.1. Let *A, B, S* and *T* be self maps of a metric space (*X, d*) and satisfying the inequality.

$$d(Ax,By) \le k \max\left\{ \frac{d(Sx,Ax)[1+d(Ty,By)]}{1+d(Sx,ty)}, \frac{d(Ty,By)[1+d(Sx,Ax)]}{1+d(Sx,Ty)}, \frac{d(Sx,Ax)[1+d(Sx,Ty)]}{1+d(Ty,By)}, \frac{d(Sx,Ty)[1+d(Sx,Ax)]}{1+d(Ty,By)}, \frac{1}{2}[d(Sx,Ax)+d(Ty,By)], \frac{1}{2}[d(Sx,By)+d(Ty,Ax)], d(sx,Ty) \right\} \quad (3.1)$$

for all $x, y \in x$, where k ≥ 0 and k < 1. Then the pairs (A,S) and (B,T) have common point of coincidence in X if and only if $C(A,S) \neq \phi$ and $C(B,T) \neq \phi$.

Prof: If part: It is trivial

Only if part: Assume $C(A,S) \neq \phi$ and $C(B,T) \neq \phi$.

Then there is a $u \varepsilon C(A,S)$ and $v \varepsilon C(B,T)$ such that

$$Au = Su = p \text{ (say)} \tag{3.2}$$

$$Bv = Tv = q \quad \text{(say)} \tag{3.3}$$

on taking $x = u$ and $y = v$ in (3.1), we get

$$d(Au,Bv) \leq k \max \Big\{ \frac{d(su,Au)[1+d(tv,Bv)]}{1+d(Su,tv)}, \frac{d(Tv,Bv)[1+d(Su,Au)]}{1+d(Su,Tv)}, \frac{d(Su,Au)[1+d(Su,Tv)]}{1+d(Tv,Bv)}, \frac{d(Su,Tv)[1+d(Su,Au)]}{1+d(Tv,Bv)}, \frac{1}{2}[d(Su,Au) + d(Tv,Bv)], \frac{1}{2}[d(Su,Bv) + d(Tv,Au)], d(Su,Tv) \Big\}$$

Using (3.2) and (3.3), we get

d(p,q) ≤ k d(p,q), a contradiction. Thus $p = q$

Therefore A, B, S and T have common point of coincidence in X.

In The proposition (2.1) of Babu et al. [9], we can obtain some more conclusions of in their paper. Therefore our result improves and strengthen proposition 3.1 and subsequent theorems in metric spaces.

Proposition 3.2: Let *A, B, S* and *T* be four self maps of a metric space (X,d) satisfying the inequality (3.1). Suppose that either

(i) $B(X) \subseteq S(X)$, the pair (B,T) satisfies property (E.A.) and $T(X)$ is a closed subspace of *X*; or

(ii) $A(X) \subseteq T(X)$, the pair (A,S) satisfies property $(E.A)$ and $S(X)$ is a closed subspace of X holds.

Then the pair (A,S) and (B,T) are satisfies the common property (*E*.A), also both the pairs (A,S) and (B,T) have common point of coincidence in *X*.

We have shorten the proof of theorem 2.2 of [9] by relaxing many lines:

Theorem 3.3: (Improved version of theorem 2.2of [9])

Let A, B, S and *T* are satisfying all the conditions given in proposition 3.2 with the following additional assumption.

The pairs (A,S) and (B,T) are owc on *X*.

Then A, B, S and T have a unique common fixed point in *X*.

Proof: By proposition 3.2 the pairs (A,S) and (B,T) have common point of coincidence. Therefore there is $u \in C(A,S)$ and $v \in C(B,T)$ such that

$$Au = Su = z \text{ (say)} = Bv = Tv \tag{3.4}$$

Now, we show that *z* is unique common point of coincidence of the pairs (A,S) and (B,T).

Let if possible *z'* is another point of coincidence of A, B, S and *T*. Then there is $u' \varepsilon C(A,S)$ and $v' \varepsilon C(B,T)$ such that

$$Au' = Su' = z' \text{ (say)} = Bv' = Tv' \tag{3.5}$$

Putting $x = u$ and $y = v'$ in inequality (3.1), we have

d(Au,Bv') ≤ k max { $\frac{d(Su,Au)[1+d(Tv',Bv')]}{1+d(Su,Tv')}$, $\frac{d(Tv',Bv')[1+d(Su,Au)]}{1+d(Su,Tv')}$

, $\frac{d(Su,Au)[1+d(Su,Tv')]}{1+d(Tv',Bv')}$, $\frac{d(Su,Tv')[1+d(Su,Au)]}{1+d(Tv',Bv')}$, $\frac{1}{2}[d(Su,Au) + d(Tv',Bv')]$, $\frac{1}{2}[d(Su,Bv') + d(Tv',Au)]$,d(Su,Tv') }

Now using (3.4) and (3.5), we get

d(z,z') ≤ k d(z,z') , and arrive at a contradiction. Hence $z = z'$ and we have $C(A,S) = \{z\} = C(B,T)$. By proposition 2.9, z is the unique common fixed point of A, B, S and *T* in *X*.

Remark 3.4: Proposition 2.5 of [9] and theorem 2.6 of [9] are remain true, if we replace completeness of *S*(*X*) and *T*(*X*) by the completeness of $S(X) \cap T(X)$ in X. For this we have given an example 2.7 in the following manner without proof.

Now we rewriting the proposition 2.5 and theorem 2.6 of [9]

Proposition 3.5: Let *A, B, S* and *T* be four self maps of a metric. Space (*X, d*) satisfying the inequality (3.1) of proposition 3.1. Suppose that (A,S) and (B,T) satisfy a common property $(E.A)$ and $S(X) \cap T(X)$ are closed subset of *X*, then *A, B, S* and *T* have unique common point of coincidence. Theorem 3.6. In addition to the above proposition 3.5 on *A, B, S* and *T*, if both the pairs (*A, S*) and (*B, T*) are owc mapson *X*, then the point of coincidence is a unique common point of A, B, S and *T*.

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