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Relative Annihilators in an ADL

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Abstract: *In this paper, we introduce the concept of relative annihilators in an ADL and study some of their properties. Also, we characterize a normal ADL R in terms of relative annihilators.*

Keywords: *Almost Distributive Lattice (ADL), relative annihilators, Normal ADL.*

1. Preliminaries

An Almost Distributive Lattice (ADL) is an algebra $(R,\vee,\wedge)$ of type (2, 2) satisfying

1. $(x\vee y)\wedge z=(x\wedge z)\vee(y\wedge z)$
2. $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$
3. $(x\vee y)\wedge y=y$
4. $(x\vee y)\wedge x=x$
5. $x\vee(x\wedge y)=x$

It can be seen directly that every distributive lattice is an ADL. If there is an element $0\in R$ such that $0\wedge a = 0$ for all $a\in R,$ then $(R,\vee,\wedge,0)$ is called an ADL with 0. As usual, an element $m\in R$ is called maximal if it is maximal element in the partially ordered set $(R,\leq)$. That is, for any $a\in R,$ $m\leq a\Rightarrow m=a.$

Let R be an ADL and $m\in R.$ Then the following are equivalent.

1). m is maximal with respect to $\leq$.

2). $m\vee a=m,$ for all $a\in R.$

3). $m\wedge a=a,$ for all $a\in R.$

4). $a\vee m$ is maximal, for all $a\in R.$

An ADL R is relatively complemented if every interval in R is a complemented lattice.

1.1. Theorem : [4] An ADL R is normal if and only if every prime ideal of R contains a unique minimal prime ideal of R.

1.2. Theorem: [5] An ADL R is normal if and only if $R=(x)^{*}\vee(y)^{*}.$

Note that, throughout this paper the letter R stands for an ADL $(R,\vee,\wedge,0).$

2. Relative Annihilators

Mark Mandelker[1] introduced relative annihilators in lattices. In this section we define a relative annihilator in an ADL and study some of its properties.

Now, we begin this section with the following definition.

2.1. Definition:

1) Let R be an ADL and A be a nonempty subset of R. For any $x \in R$, we define $x \wedge A = \{x \wedge a \mid a \in A\}$.

2) For any subsets A, B of an ADL R, we define $\lfloor A, B \rfloor = \{x \in R \mid x \wedge A \subseteq B\}$.

The following lemma can be verified routinely.

2.2. Lemma : Let A, B, C be any three subsets of an ADL R.

Then 1). $A \subseteq B \Rightarrow \lfloor C, A \rfloor \subseteq \lfloor C, B \rfloor$

2). $A \subseteq B \Rightarrow \lfloor B, C \rfloor \subseteq \lfloor A, C \rfloor$

3). $\lfloor A, B \rfloor \cap \lfloor A, C \rfloor = \lfloor A, B \cap C \rfloor$

4). $\lfloor A, B \rfloor \cup \lfloor A, C \rfloor \subseteq \lfloor A, B \cup C \rfloor$

5). $\lfloor A, C \rfloor \cup \lfloor B, C \rfloor \subseteq \lfloor A \cap B, C \rfloor$

6). $\lfloor A, C \rfloor \cap \lfloor B, C \rfloor = \lfloor A \cup B, C \rfloor$

In general for any family $\{A_\alpha \mid \alpha \in \Delta\}$ of subsets of R, $\lfloor \bigcup_{\alpha \in \Delta} A_\alpha, C \rfloor = \bigcap_{\alpha \in \Delta} \lfloor A_\alpha, C \rfloor$

In general the set $\lfloor A, B \rfloor$ is not an ideal of R. In the following example we show that the set $\lfloor A, B \rfloor$ is not an ideal when A, B are subsets of an ADL R.

2.3. Example : Let $R = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on R as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then $(R, \vee, \wedge, 0)$ is an ADL with 0.

Let $A = \{a, b\}$ and $B = \{0, a, b\}$ be two subsets of R. For $b \in B$ and $c \in R$, we have $b \wedge c = c \notin B$. Therefore B is not an ideal of R. Also $\lfloor A, B \rfloor = \{0, a, b\}$.

Therefore $\lfloor A, B \rfloor = B$. Since B is not an ideal, we have $\lfloor A, B \rfloor$ is not an ideal of R. Thus, when A, B are any two subsets of R, the set $\lfloor A, B \rfloor$ is not an ideal of R.

In the following result, we prove that $\lfloor A, B \rfloor$ is an ideal of R if B is an ideal of R.

2.4. Theorem : If A is a subset of an ADL R and B is an ideal in R, then $\lfloor A, B \rfloor$ is an ideal of R and $B \subseteq \lfloor A, B \rfloor$

Proof : Clearly, $0 \in \lfloor A, B \rfloor$ Therefore $\lfloor A, B \rfloor$ is non-empty. Let $x, y \in \lfloor A, B \rfloor$. Then $x \wedge a \in B$ and $y \wedge a \in B$ for every $a \in A$. Since B is an ideal, $(x \wedge a) \vee (y \wedge a) \in B$ for all $a \in A$. That is $(x \vee y) \wedge a \in B$, for all $a \in A$. Therefore $x \vee y \in \lfloor A, B \rfloor$. Let $x \in \lfloor A, B \rfloor$. Then $x \wedge a \in B$, for all

$a \in A$. Since B is an ideal of R, for any $r \in R$, $r \wedge x \wedge a \in B$ and hence $x \wedge r \wedge a \in B$, for every $a \in A$. This gives $x \wedge r \in \lfloor A, B \rfloor$. Therefore $\lfloor A, B \rfloor$ is an ideal of R. Now, since B is an ideal, $x \in B \Rightarrow x \wedge a \in B,$ for all $a \in A$.

This gives $x \wedge A \subseteq B$ and hence $x \in \lfloor A, B \rfloor$. Therefore $B \subseteq \lfloor A, B \rfloor$

2.5. Definition : Let A be a subset of an ADL R and B is an ideal in R. Then we call the ideal $\lfloor A, B \rfloor$ as a relative annihilator of A with respect to B.

By usual verification we get the following :

1. If $B = \{ 0 \}$, then $\lfloor A, B \rfloor$ is an annihilator of A.
2. If $A \subseteq B$ and B is an ideal of R then $\lfloor A, B \rfloor = R$.
3. If $A = \{0\}$ and B is an ideal of R then $\lfloor A, B \rfloor = R$

2.6. Definition : Let R be an ADL and $a, b \in R.$ Then we define

$$\lfloor a, b \rfloor = \{x \in R \mid x \wedge a = b \wedge x \wedge a\}.$$

Observe that $x \in \lfloor a, b \rfloor \Leftrightarrow x \wedge a = b \wedge x \wedge a \Leftrightarrow b = b \vee (x \wedge a)$

Now we prove the following.

2.7. Lemma : For any $a, b \in R$, $\lfloor a, b \rfloor = \lfloor (a], (b] \rfloor$

Proof : Let $x \in \lfloor a, b \rfloor$. Then $x \wedge a = b \wedge x \wedge a$ and for any $t \in R, x \wedge a \wedge t = b \wedge x \wedge a \wedge t$. Clearly, $x \wedge a \wedge t \in (b]$. Therefore $x \wedge s \in (b]$, for every $s = a \wedge t \in (a]$ and hence $x \in \lfloor (a], (b] \rfloor$

Thus $\lfloor a, b \rfloor \subseteq \lfloor (a], (b] \rfloor$

Let $x \in \lfloor (a], (b] \rfloor$. Then $x \wedge s \in (b],$ for all $s \in (a]$. Since $a \in (a],$ we get $x \wedge a \in (b]$

and hence $b \wedge x \wedge a = x \wedge a.$ Therefore $x \in \lfloor a, b \rfloor$. This gives $\lfloor (a], (b] \rfloor \subseteq \lfloor a, b \rfloor$.

Hence $\lfloor a, b \rfloor = \lfloor (a], (b] \rfloor$

By Theorem 1.4 and from Lemma 1.7, we get the following.

2.8. Corollary: For any $a, b \in R,$ $\lfloor a, b \rfloor$ is an ideal of R. $\lfloor t, a \rfloor$

2.9. Lemma : Let $c \in R$ and A be an ideal of R.

Then for any $t \in (c]$ and $a \in A,$ $\lfloor c, a \rfloor \subseteq \lfloor t, a \rfloor \subseteq \lfloor (t], (a] \rfloor$

Proof : Let A be any ideal of R and $c \in R.$ Let $t \in (c]$.

Then $(t] \subseteq (c]$. and hence from (2) of Lemma 1.2, we get $\lfloor (c], (a] \rfloor \subseteq \lfloor (t], (a] \rfloor$ ....(I)

Since A is an ideal and $a \in A,$ we get $(a] \subseteq A$

Therefore from (1) of Lemma 1.2, we get $\lfloor (t], (a] \rfloor \subseteq \lfloor (t], A \rfloor$ ....(II)

From (I) and (II), we get $\lfloor (c], (a] \rfloor \subseteq \lfloor (t], (a] \rfloor \subseteq \lfloor (t], A \rfloor$

Therefore from Lemma 1.7, we get $\lfloor c,a \rfloor \subseteq \lfloor t,a \rfloor \subseteq \lfloor (t],A \rfloor$

The following lemma can be verified routinely.

2.10. Lemma : For any $a,b \in R,$

1). $\lfloor a,b \rfloor = \lfloor b,a \rfloor$ if and only if $a \wedge b = b$ and $b \wedge a = a.$

2). $\lfloor a,0 \rfloor = \lfloor 0,a \rfloor$ if and only if $a = 0.$

3). For any $a,b \in R-\{0\}$, $\lfloor a,b \rfloor = \lfloor b,a \rfloor$ if and only if R is a discrete ADL.

Now, we prove some important properties of relative annihilators.

2.11. Theorem : Let R be an ADL and $a,b \in R.$ Then

1). $s \in \lfloor a,b \rfloor \Leftrightarrow a \in \lfloor s,b \rfloor$

2). $s \in \lfloor a,b \rfloor \Rightarrow \lfloor a,s \rfloor \subseteq \lfloor a,b \rfloor$

3). For any $a,b \in R$, $a \in \lfloor b,a \wedge b \rfloor$ and $b \in \lfloor a,b \rfloor$

Proof : 1) Let $a,b \in R.$ Then $s \in \lfloor a,b \rfloor \Rightarrow s \wedge a = b \wedge s \wedge a$

$\Rightarrow s \wedge a \wedge s = b \wedge s \wedge a \wedge s$

$\Rightarrow a \wedge s = b \wedge a \wedge s$

$\Rightarrow a \in \lfloor s,b \rfloor$

Similarly, we can prove $a \in \lfloor s,b \rfloor \Rightarrow s \in \lfloor a,b \rfloor$

2) Let $s \in \lfloor a,b \rfloor$. Then $s \wedge a = b \wedge s \wedge a.$

Now $x \in \lfloor a,s \rfloor \Rightarrow x \wedge a = s \wedge x \wedge a$

$\Rightarrow x \wedge a = x \wedge s \wedge a$

$\Rightarrow x \wedge a = x \wedge b \wedge s \wedge a$ (since $s \wedge a = b \wedge s \wedge a$)

$\Rightarrow x \wedge a = b \wedge s \wedge x \wedge a$

$\Rightarrow x \wedge a = b \wedge x \wedge a$ (since $s \wedge x \wedge a = x \wedge a$)

$\Rightarrow x \in \lfloor a,b \rfloor$ Therefore $\lfloor a,s \rfloor \subseteq \lfloor a,b \rfloor$

3) is clear.

2.12. Lemma : For any a,b,c in an ADL R, we get the following.

1). If $a \leq b$ then for any $c \in R,$ $\lfloor b,c \rfloor \subseteq \lfloor a,c \rfloor$ and $\lfloor c,a \rfloor \subseteq \lfloor c,b \rfloor$

2). $\lfloor a,b \rfloor = \lfloor a,a \wedge b \rfloor = \lfloor a,b \wedge a \rfloor = \lfloor a \vee b,a \rfloor = \lfloor b \vee a,a \rfloor = \lfloor a \vee b,a \wedge b \rfloor$

3). $R = \lfloor 0,a \rfloor = \lfloor a,a \rfloor = \lfloor a,a \vee b \rfloor = \lfloor a,b \vee a \rfloor = \lfloor a \wedge b,a \rfloor = \lfloor b \wedge a,a \rfloor = \lfloor a \wedge b,a \vee b \rfloor$

4). For any $a,b,c \in R$, $\lfloor a \vee b,c \rfloor = \lfloor b \vee a,c \rfloor$, $\lfloor a \wedge b,c \rfloor = \lfloor b \wedge a,c \rfloor$

$\lfloor c,a \wedge b \rfloor = \lfloor c,b \wedge a \rfloor$ and $\lfloor c,a \vee b \rfloor = \lfloor c,b \vee a \rfloor$

5). For any $a, b, c \in R$, i). $\lfloor a, c \rfloor \vee \lfloor b, c \rfloor \subseteq \lfloor a \wedge b, c \rfloor$

ii). $\lfloor a, b \rfloor \vee \lfloor a, c \rfloor \subseteq \lfloor a, b \vee c \rfloor$

iii). $\lfloor a \vee b, c \rfloor = \lfloor a, c \rfloor \cap \lfloor b, c \rfloor$

iv). $\lfloor a, b \wedge c \rfloor = \lfloor a, b \rfloor \cap \lfloor a, c \rfloor$

6). In addition to these, if R is a relatively complemented ADL then

$$\lfloor a, b \rfloor \vee \lfloor a, c \rfloor = \lfloor a, b \vee c \rfloor$$

Proof :

1). Let a, b be any two elements of R such that $a \leq b.$ Then $a \wedge b = a = b \wedge a.$

Now, $x \in \lfloor b, c \rfloor \Rightarrow x \wedge b = c \wedge x \wedge b$

$$\Rightarrow x \wedge b \wedge a = c \wedge x \wedge b \wedge a$$

$$\Rightarrow x \wedge a = c \wedge x \wedge a \Rightarrow x \in \lfloor a, c \rfloor$$

Therefore $\lfloor b, c \rfloor \subseteq \lfloor a, c \rfloor$

Now, $x \in \lfloor c, a \rfloor \Rightarrow x \wedge c = a \wedge x \wedge c \Rightarrow x \wedge c = b \wedge a \wedge x \wedge c$

$\Rightarrow x \wedge c = b \wedge x \wedge c$ (since $x \wedge c = a \wedge x \wedge c$)

$\Rightarrow x \in \lfloor c, b \rfloor$. Therefore $\lfloor c, a \rfloor \subseteq \lfloor c, b \rfloor$

2). Let a, b be any two elements of R.

Now, $x \in \lfloor a, b \rfloor \Leftrightarrow x \wedge a = b \wedge x \wedge a \Leftrightarrow x \wedge a = a \wedge b \wedge x \wedge a \Leftrightarrow x \in \lfloor a, a \wedge b \rfloor$

Therefore $\lfloor a, b \rfloor = \lfloor a, a \wedge b \rfloor$

Similarly, we can prove the remaining results.

3) is clear.

4). Let a, b, c be any three elements of R. $x \in \lfloor a \vee b, c \rfloor$

Now, $x \in \lfloor a \vee b, c \rfloor \Rightarrow x \wedge (a \vee b) = c \wedge x \wedge (a \vee b)$

$$\Rightarrow x \wedge (a \vee b) \wedge (b \vee a) = c \wedge x \wedge (a \vee b) \wedge (b \vee a)$$

$$\Rightarrow x \wedge (b \vee a) \wedge (b \vee a) = c \wedge x \wedge (b \vee a) \wedge (b \vee a)$$

$$\Rightarrow x \wedge (b \vee a) = c \wedge x \wedge (b \vee a)$$

Therefore $\lfloor a \vee b, c \rfloor \subseteq \lfloor b \vee a, c \rfloor$ Similarly, we can prove that $\lfloor b \vee a, c \rfloor \subseteq \lfloor a \vee b, c \rfloor$

Hence $\lfloor a \vee b, c \rfloor = \lfloor b \vee a, c \rfloor$

Similarly, we can prove the remaining results.

5). Let a, b, c be any three elements of R.

Now, from (1), $a \wedge b \leq b \Rightarrow \lfloor b, c \rfloor \subseteq \lfloor a \wedge b, c \rfloor$

Similarly, we get $b \wedge a \le a \Rightarrow \lfloor a,c \rfloor \subseteq \lfloor b \wedge a,c \rfloor = \lfloor a \wedge b,c \rfloor$ [from (4)].

Therefore $\lfloor a,c \rfloor \vee \lfloor b,c \rfloor \subseteq \lfloor a \wedge b,c \rfloor$

(ii) : Proof is similar to (i). $\lfloor a \vee b,c \rfloor \subseteq \lfloor a,c \rfloor \cap \lfloor b,c \rfloor$

(iii): Let $a,b,c \in R$. Then from (1) and (4), we get $\lfloor a \vee b,c \rfloor \subseteq \lfloor a,c \rfloor$ $x \in \lfloor b,c \rfloor$

Now, $x \in \lfloor a,c \rfloor \cap \lfloor b,c \rfloor \Rightarrow x \in \lfloor a,c \rfloor$ and $x \in \lfloor b,c \rfloor$

$$\Rightarrow x \wedge a = c \wedge x \wedge a \text{ and } x \wedge b = c \wedge x \wedge b$$

$$\Rightarrow (x \wedge a) \vee (x \wedge b) = (c \wedge x \wedge a) \vee (c \wedge x \wedge b)$$

$$\Rightarrow x \wedge (a \vee b) = c \wedge x \wedge (a \vee b)$$

$$\Rightarrow x \in \lfloor a \vee b,c \rfloor$$

Therefore we get $\lfloor a,c \rfloor \cap \lfloor b,c \rfloor \subseteq \lfloor a \vee b,c \rfloor$

Hence $\lfloor a \vee b,c \rfloor = \lfloor a,c \rfloor \cap \lfloor b,c \rfloor$

(iv): Let $a,b,c \in R$. Then from (1) and (4), we get $\lfloor a,b \wedge c \rfloor \subseteq \lfloor a,b \rfloor \cap \lfloor a,c \rfloor$

Now, $x \in \lfloor a,b \rfloor \cap \lfloor a,c \rfloor \Rightarrow x \in \lfloor a,b \rfloor$ and $x \in \lfloor a,c \rfloor$

$$\Rightarrow x \wedge a = b \wedge x \wedge a \text{ and } x \wedge a = c \wedge x \wedge a$$

$$\Rightarrow (x \wedge a) \wedge (x \wedge a) = b \wedge x \wedge a \wedge c \wedge x \wedge a$$

$$\Rightarrow x \wedge a = b \wedge c \wedge x \wedge a$$

$$\Rightarrow x \in \lfloor a,b \wedge c \rfloor$$

Therefore we get $\lfloor a,b \rfloor \cap \lfloor a,c \rfloor \subseteq \lfloor a,b \wedge c \rfloor$

Hence $\lfloor a,b \wedge c \rfloor = \lfloor a,b \rfloor \cap \lfloor a,c \rfloor$

6). Let R be a relatively complemented ADL and $a,b,c \in R$. From (5), we have $\lfloor a,b \rfloor \vee \lfloor a,c \rfloor \subseteq \lfloor a,b \vee c \rfloor$. Now, let $x \in \lfloor a,b \vee c \rfloor$ Consider the interval $[0, a \vee x]$. Since R is relatively complemented ADL, every interval in R is a complemented lattice. Therefore $[0, a \vee x]$ is a complemented lattice. Let a' be the complement of a in the interval $[0, a \vee x]$. Then $a \wedge a' = 0$ and $a \vee a' = a \vee x$. Now,

$$x \in \lfloor a,b \vee c \rfloor \Rightarrow (b \vee c) \wedge x \wedge a = x \wedge a$$

$$\Rightarrow a' \vee [(b \vee c) \wedge x \wedge a] = a' \vee (x \wedge a)$$

$$\Rightarrow (a' \vee b \vee c) \wedge (a' \vee x) \wedge (a' \vee a) = (a' \vee x) \wedge (a' \vee a)$$

$$\Rightarrow (a' \vee b \vee c) \wedge (a' \vee x) \wedge (a \vee x) = (a' \vee x) \wedge (a \vee x)$$

$$\Rightarrow (a' \vee b \vee c) \wedge (a' \vee x) \wedge (a \vee x) \wedge x = (a' \vee x) \wedge (a \vee x) \wedge x$$

$$\Rightarrow (a' \vee b \vee c) \wedge x = x$$

$$\Rightarrow [(a' \vee b) \wedge x] \vee [(a' \vee c) \wedge x] = x$$

Now, $(a'\vee b)\wedge x\wedge a \;=\; (a'\wedge x\wedge a)\vee(b\wedge x\wedge a)$

$$= 0\vee(b\wedge x\wedge a)$$

$$= (a'\vee b)\wedge b\wedge x\wedge a$$

$$= b\wedge(a'\vee b)\wedge x\wedge a \quad (\text{since } (a'\vee b)\wedge b = b)$$

This gives $(a'\vee b)\wedge x\in\lfloor a,b\rfloor$. Similarly, we get $(a'\vee c)\wedge x\in\lfloor a,c\rfloor$

Therefore $x=[(a'\vee b)\wedge x]\vee[(a'\vee c)\wedge x] \in\lfloor a,b\rfloor\vee\lfloor a,c\rfloor$

This gives $\lfloor a,b\vee c\rfloor \subseteq\lfloor a,b\rfloor\vee\lfloor a,c\rfloor$. Hence $\lfloor a,b\rfloor\vee\lfloor a,c\rfloor = \lfloor a,b\vee c\rfloor$

2.13. Lemma : For any a,b,c in an ADL R, we get the following.

1). $\lfloor a,0\rfloor=(a)^*$, where $(a)^*=\{x\in R\mid a\wedge x=0\}$

2). If $a\vee b$ or $b\vee a$ is an element of $\lfloor a,b\rfloor$ then $\lfloor a,b\rfloor=R$

3). If $a\wedge b=0,$ then for any $c\in R,$ $a\in\lfloor b,c\rfloor$

Let $a,b\in R$ with $a<b$ and $x,y\in[a,b].$ Then we can observe that $\lfloor x,a\rfloor\cap[a,b]$ is an ideal in $[a,\ b]$.

Now we prove the following theorem.

2.14. Theorem : Let Let I,J be any two ideals of R and $x,y\in R.$ Then

1). $\lfloor I,(y]\rfloor = \bigcap_{x\in I}\lfloor x,y\rfloor$

2). $\lfloor I,\ J\rfloor = \bigcap_{x\in I}\lfloor (x],J\rfloor$

3). $\lfloor (x],J\rfloor = \lfloor x,y\rfloor = \mathrm{V}_{y\in J}\lfloor x,y\rfloor$.

4). Let $a,b\in R$ with $a<b$ and $x,y\in[a,b].$ Then

$$\{\lfloor x,a\rfloor\vee\lfloor y,a\rfloor\}\cap[a,b] = \{\lfloor x,a\rfloor\cap[a,b]\}\vee\{\lfloor y,a\rfloor\cap[a,b]\}.$$

Proof :

(1): For any $x\in I,(x]\subseteq I$. Therefore from 2 of Lemma 1.2,

we get $\lfloor I,(y]\rfloor\subseteq\lfloor (x],(y]\rfloor = \lfloor x,y\rfloor$. Thus $\lfloor I,(y]\rfloor\subseteq\bigcap_{x\in I}\lfloor x,y\rfloor$.

Again, let $t\in\lfloor x,y\rfloor$ for all $x\in I.$ Then $t\wedge x=y\wedge t\wedge x\in(y]$ for all $x\in I.$

This gives $t\in\lfloor I,(y]\rfloor$. Therefore $\bigcap_{x\in I}\lfloor x,y\rfloor\subseteq\lfloor I,(y]\rfloor$.

(2) Take $J=(y]$ in the above result (1).

(3) : From (1) of Lemma 1.2 and from Lemma 1.7, for any $y\in J$,

we have $\lfloor x,y\rfloor = \lfloor (x],(y]\rfloor \subseteq\lfloor (x],J\rfloor$. Let $t\in\lfloor (x],J\rfloor$. Then $t\wedge s\in J$ for all $s\in(x]$. Since $x\in(x]$, we get $t\wedge x=y$ for some $y\in J$. Clearly, $y\wedge t\wedge x=t\wedge x$. Therefore $t\in\lfloor x,y\rfloor$ for some $y\in J$. This gives $\lfloor (x],J\rfloor\subseteq\bigvee_{y\in J}\lfloor x,y\rfloor$.

Therefore $\lfloor (x], J \rfloor = \bigvee_{y \in J} \lfloor x, y \rfloor$.

(4): Let $a, b \in R$ with $a < b$ and $x, y \in [a, b]$.

Clearly, $\{\lfloor x, a \rfloor \cap [a, b]\}$ $\vee$ $\{\lfloor y, a \rfloor \cap [a, b]\} \subseteq \{\lfloor x, a \rfloor \vee \lfloor y, a \rfloor\} \cap [a, b]$.

Now, let $s \in \{\lfloor x, a \rfloor \vee \lfloor y, a \rfloor\} \cap [a, b]$.

Then $s = t_1 \vee t_2$, where $t_1 \in \lfloor x, a \rfloor$, $t_2 \in \lfloor y, a \rfloor$ and $s \in [a, b]$.

This gives $a \le s = t_1 \vee t_2 \le b$ where $t_1 \wedge x = a \wedge t_1 \wedge x$ and $t_2 \wedge y = a \wedge t_2 \wedge y$.

Now $s = (a \vee s) \wedge b = (a \vee t_1 \vee t_2) \wedge b = [(a \vee t_1) \vee (a \vee t_2)] \wedge b = [(a \vee t_1) \wedge b] \vee [(a \vee t_2) \wedge b]$.

Clearly, $(a \vee t_1) \wedge b$, $(a \vee t_2) \wedge b \in [a, b]$.

Now we prove that $(a \vee t_1) \wedge b \in \lfloor x, a \rfloor$ and $(a \vee t_2) \wedge b \in \lfloor y, a \rfloor$.

$$
\begin{aligned}
\text{Now,}\quad (a \vee t_1) \wedge b \wedge x &= (a \wedge b \wedge x) \vee (t_1 \wedge b \wedge x) \\
&= (a \wedge b \wedge x) \vee [b \wedge (t_1 \wedge x)] \\
&= (a \wedge b \wedge x) \vee [b \wedge (a \wedge t_1 \wedge x)] \\
&= (a \wedge b \wedge x) \vee (t_1 \wedge a \wedge b \wedge x) \\
&= (a \wedge b \wedge x) \\
&= a \wedge (a \vee t_1) \wedge b \wedge x
\end{aligned}
$$

Therefore $(a \vee t_1) \wedge b \in \lfloor x, a \rfloor$. Similarly, we can prove $(a \vee t_2) \wedge b \in \lfloor y, a \rfloor$.

Therefore we get $s \in \{\lfloor x, a \rfloor \cap [a, b]\} \{\lfloor y, a \rfloor \cap [a, b]\}$. This proves the result.

3. Characterization of Normal ADLs in terms of Relative annihilators

In this section, we characterize a normal ADL in terms of Relative annihilators.
First we prove the following Lemma.

3.1. Lemma : Let $a, b \in R$. Then $x \in \lfloor a, b \rfloor$ if and only if $a \wedge x \le b \wedge x$.

Proof : Let a, b be any two elements of R. Assume that $x \in \lfloor a, b \rfloor$. Then $x \wedge a = b \wedge x \wedge a$. Now $x \wedge a \wedge x = b \wedge x \wedge a \wedge x$. This gives $a \wedge x = a \wedge b \wedge x \le b \wedge x$. Therefore $a \wedge x \le b \wedge x$. Conversely, assume that $a \wedge x$ and $b \wedge x$ are comparable. Without loss of generality, take $a \wedge x \le b \wedge x$. Then $a \wedge x = a \wedge x \wedge b \wedge a$. This gives $a \wedge x \wedge a = a \wedge x \wedge b \wedge x \wedge a$ and hence $x \wedge a = b \wedge x \wedge a$. Therefore $x \in \lfloor a, b \rfloor$

3.2. Lemma : Let I be any ideal of an ADL R and for any prime filter F of R, $I \cap F \neq \phi$. Then $I = R$.

3.3. Corollary : For any $a, b \in R$ and for any prime filter F of R , $\{\lfloor a, b \rfloor \vee \lfloor b, a \rfloor\} \cap F \neq \phi$

3.4. Lemma : Let F be any prime filter of R. For any $a, b \in R$, if $b \in F \vee [a)$ then $F \cap \lfloor a, b \rfloor$ is non-empty .

Proof : Let $b \in F \vee [a)$. Then $b = t \wedge s$ for some $t \in F$ and $s \in [a)$. That is $b = t \wedge (s \vee a)$ = $(t \wedge s) \vee (t \wedge a)$ = $b \vee (t \wedge a)$. This gives $b \wedge (t \wedge a) = t \wedge a$. Therefore $t \in \lfloor a, b \rfloor$ Thus $t \in F \cap \lfloor a, b \rfloor$. Therefore $F \cap \lfloor a, b \rfloor$ is non-empty.

Now, we conclude this section with the following theorem in which we characterize a normal ADL R in terms of relative annihilators.

3.5. Theorem : In an ADL R, the following are equivalent.

1). R is a normal ADL.

2). $\lfloor a,b \rfloor \vee \lfloor b,a \rfloor = R$, for any $a,b \in R,$ with $a \wedge b = 0.$

3). For any prime filter F in R and for any $a,b \in R$ with $a \wedge b = 0$, there exists $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable.

Proof : (1) $\Rightarrow$ (2): Assume that R is a normal ADL. Then from Theorem 0.2, we have every prime filter in R is contained in a unique maximal filter of R. Let $a,b \in R$ with $a \wedge b = 0$. We have to prove that $R = \lfloor a,b \rfloor \vee \lfloor b,a \rfloor$. Suppose $I = \lfloor a,b \rfloor \vee \lfloor b,a \rfloor \neq R.$ Then I is a proper ideal of R and hence it is contained in a maximal ideal, say $M.$ Write $F = R - M.$ Then F is a prime filter and $F \cap I = \emptyset.$ Now, we prove that the prime filter F is contained in two distinct maximal filters of R. Consider the filter $F \vee [a).$ If $b \in F \vee [a),$ then from Lemma 2.4, we get $F \cap (a,b) \neq \emptyset.$ This gives $F \cap I \neq \emptyset.$ This is a contradiction. Therefore $b \notin F \vee [a)$. Therefore $F \vee [a)$ is a proper filter of R. Similarly, we can prove that $F \vee [b)$ is a proper filter of R. Therefore, there exist two maximal filters G_1, G_2 in R such that $F \vee [a) \subseteq G_1$ and $F \vee [b) \subseteq G_2$. Since $a \wedge b = 0$ and $0 \notin G_1$, we get $b \notin G_1$. Hence we get $G_1 \neq G_2$.

Also, $F \subseteq G_1$ and $F \subseteq G_2$. Thus the prime filter F is contained in two distinct maximal filters G_1 and G_2. This is a contradiction. Therefore $\lfloor a,b \rfloor \vee \lfloor b,a \rfloor = R$ for any $a,b \in R$ with $a \wedge b = 0.$

(2) $\Rightarrow$ (3) : Assume the condition (2). Let F be any prime filter of R and $a,b \in R$ with $a \wedge b = 0$. Then by (2), $\lfloor a,b \rfloor \vee \lfloor b,a \rfloor = R$ Let $z \in F \subseteq R$. Then we can write $z = x \vee y$ for some $x \in \lfloor a,b \rfloor$ and $y \in \lfloor b,a \rfloor$. Since F is prime and $z = x \vee y \in F$, we get either $x \in F$ or $y \in F.$ Suppose $x \in F$. Since $x \in \lfloor a,b \rfloor$. from Lemma 2.1, we get $a \wedge x \leq b \wedge x$. Thus there is an element $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable. Similarly we get $a \wedge x$ and $b \wedge x$ are comparable, if $y \in F.$

(3) $\Rightarrow$ (1) : Assume the condition (3). We have to prove that R is normal. Let $a,b \in R$ and $a \wedge b = 0$. Now, we prove that $(a)^* \vee (b)^* = R$. Suppose $(a)^* \vee (b)^* \neq R.$ Then there exists a maximal ideal M of R such that $(a)^* \vee (b)^* \subseteq M$. Write $F = R - M.$ Then F is a prime filter. Therefore from (3), there exists $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable. Without loss of generality, suppose that $a \wedge x \leq b \wedge x$. Then $a \wedge x = a \wedge x \wedge b \wedge x = a \wedge b \wedge x = 0$. Therefore $x \in (a)^* \subseteq M$. This is a contradiction (since $x \in F$). Therefore $(a)^* \vee (b)^* = R$.

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