Structure of $u -$ Injective Hull in Ultra-Quasi-Pseudo Metric Space

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Abstract: *In this paper we establish fixed point theorem for nonexpansive mappings in u-injective ultra-quasi-pseudo-metric space. We generalize Agyingi [2] ultra-quasi-metric tight extension of ultra-quasi-metric space to the case of nonexpansive mappings in u-injective ultra-quasi-pseudo-metric space. Our results are the extension of the results obtain by [2] and also we will generalize some crucial results about tight extensions of ultra-quasi-pseudo metric spaces.*

Keywords: *u-Injective hull, q-hyperconvexity, tight extension, tight span, ultra-quasi-metric space.*

1. INTRODUCTION

The idea for injective hull of a metric space was introduced by Isbell. In [6] Isbell showed that every metric space X has an injective hull T_X which is compact if X is compact. Dress [3] presented also examine the idea of the tight extension of a metric space. It is known that Dress's theory is equivalent to the theory of the injective hull of a metric space independently discussed by Isbell some years earlier. Since one of the most interesting concepts in the theory of hyperconvex metric spaces is the notion of the injective hull of a metric space [5], it is very natural to wonder about the existence of an injective hull in the field of generalized metric spaces. From that point forward a considerable measure from claiming worth of effort have been carried in regards to those hyperconvexity for nonexpansive mappings (see [2, 4, 7, 9]).

Kunzi and Olela Otafudu [10] presented the construction of the ultra-quasi-metrically injective hull in the category of $T_0 -$ultra-quasi-metric spaces and contracting maps. Comparable studies have been done in the category of ultra-metric spaces and contracting maps by Bayod and Maurica [13].

In this paper we prepare the structure for u-injective hull of ultra-quasi-metrically by tight extension for ultra-quasi-pseudo-metric space for nonexpansive maps. The principle consequence of this paper is to the u-injective hull for ultra-quasi-metrically Toward tight extension for ultra-quasi-pseudo-metric space with nonexpansive maps. Furthermore, we examine those ultra-quasi-metrically u-injective hull from claiming an ultra-quasi-pseudo-metric space will be parallel of the q-hyperconvex hull of an ultra-quasi-pseudo-metric space.

2. PRELIMINARIES

A metric space (X, d) is called hyperconvex if any collection $\{(x_i, r_i)\}_{i \in I} \subset X \times [0, \infty)$ with the property that $d(x_i, r_i) \leq r_i + r_i$ for all pairs of indices $i, j \in I$, satisfies $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$ where $B(x, r) = \{y \in X : d(x, y) \leq r\}$. With the help of zorn's lemma**.**

Definition 2.1. Let X be set and let $u : X \times X \to [0, \infty)$ be a mapping into the set $[0, \infty)$ of nonnegative reals. Then u is an ultra-quasi-pseudo-metric space on X if

$$(a) \ u(x, x) = 0 \ \forall \ x \in X,$$

$(b) \ u(x, z) \leq \max\{u(x, y), u(y, z)\}$ whenever $x, y, z \in X$ we remark here that the conjugate u^t of u where $u^t(x, y) = u(y, x)$ whenever $x, y \in X$ is also ultra-quasi-pseudo-metric on X.

If u also satisfies the condition,

(c) for any $x, y \in X, u(x, y) = 0 = u(y, x)$ implies that $x = y$, then u is called ultra-quasi-pseudo-metric on X.

Notice that $u^s = \max\{u, u^t\} = u \vee u^t$ is ultrametric on X

Definition 2.2. A map $f: (X, d_X) \to (Y, d_Y)$ between two ultra-quasi-pseudo-metric space (X, d_X) and (Y, d_Y) is called nonexpansive if $d_Y(f(x), f(y)) \leq d_X(x, y)$ holds for any

$$x, y \in X$$

Corollary 2.3 (see [10]). Let (X, u) be an ultra-quasi-pseudometric space. Consider $f: X \to [0, \infty)$ and let $x, y \in X$. Then the following are equivalent:

(a) $n\big(f(x), f(y)\big) \leq u(x, y)$;

(b) $f(x) \leq \max\{f(y), u(x, y)\}$.

Corollary 2.4 (see [10]). Let (X, u) be an ultra-quasi-pseudo-metric space.

(a) Then $f: (X, u) \to ([0, \infty), n)$ is a nonexpansive map if and only if

$f(x) \leq \max\{f(y), u(x, y)\}$ whenever $x, y \in X$.

(b) Then $f: (X, u) \to ([0, \infty), n^{-1})$ is a nonexpansive map if and only if

$f(x) \leq \max\{f(y), u(y, x)\}$ whenever $x, y \in X$.

3. Q-HYPERCONVEX HULLS OF ULTRA-QUASI-PSEUDO-METRIC SPACES

Next we recall some basic concepts mainly from [10,13] belonging to the theory of the q-hyperconvex hull of an ultra-quasi-pseudo-metric space.

Let (X, d) be a ultra-quasi-pseudo-metric space. We shall say that a function pair $f = (f_1, f_2)$ on (X, d) where $f_i : X \to [0, \infty)(i = 1, 2)$ is ample provided that $d(x, y) \leq \max\{f_2(x) + f_1(y)\}$ whenever $x, y \in X$.

Let us denote by uP_X the set of all ultra-ample function pairs on ultra-quasi-pseudo-metric space (X, d). For each $f, g \in P_X$,

$$N(f, g) = \max\{sup_{x \in X} n\left(f_1(x), g_1(x)\right), sup_{x \in X} n\left(g_2(x), f_2(x)\right)\},$$

Then N is an extend ultra-quasi-pseudo-metric on P_X . we recall that (Q_X, D) is said to be the q-hyperconvex hull of (X, d). A ultra-quasi-pseudo-metric space X is said to be q-hyperconvex if $f \in Q_X$ implies that there is an $x \in X$ such that $f = f_x$.

Definition 3.1. A quasi-pseudo-metric space (X, d) is called q-hyperconvex that for each family $(x_i)_{i \in I}$ of points in X and families $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ of nonnegative real numbers satisfying $d(x_j, x_j) \leq r_i + s_j$ whenever $i, j \in I$, the following condition holds:

$$\bigcap_{i \in I}(C_d(x_i, r_i) \cap C_{d^t}(x_i, s_i)) \neq \emptyset$$

4. INJECTIVE HULL OF ULTRA-QUASI-PSEUDO METRIC TIGHT EXTENSIONS

In this section we will generalize some crucial results about tight extensions of metric spaces from [5] to our injective hull in ultra-quasi-pseudo metric setting as defined below. Herrlich [7] prove that hyperconvex hull is equal to injective hull. It will be not difficult to substantiate proportionality the middle of Injectivity What's more Hyperconvexity.

Definition 4.1. [6] A metric space (X, d) is said to be injective if it has the following extension property whenever Z is a subspace of a metric space Y and $f: Z \to X$ is nonexpansive, then f has a nonexpansive extension $\bar{f} : Y \to X$.

Definition 4.2. Let X be a subspace of q-hyperconvex-ultra-quasi-pseudo-metric space (Y, d_Y). Then (Y, d_Y) is called tight extension of X if for any ultra-quasi-pseudo-metric ρ on Y that satisfies $\rho \leq d_Y$ and agrees with d_Y on $X \times X$, we have that $\rho = d_Y$.

Remark 4.3. For any $q-$hyperconvex-ultra-quasi-pseudo-metric tight extension Y_1 of X, any $q-$hyperconvex-ultra-quasi-pseudo metric extension (Y_2, d) of X and any nonexpansive map

$\varphi: Y_1 \to Y_2$ satisfying $\varphi(x) = x$ whenever $x \in X$.

It was shown in [8] that the map $e_X: (X, d) \to (Q_X, N)$ from ultra-quasi-pseudo metric space (X, d) to its ultra-quasi-metrically injective hull (Q_X, N) defined by $e_X(a) = f_a$ whenever $a \in X$ is an isometric embedding. We will proceed now with the help of (lemma 4.4) to show that Q_X is tight extension of $e_X(X)$ with quasi metrically injective hull.

Lemma 4.4 (see [3], Remark 7). Let (X, u) be a u-injective hull of ultra-quasi-pseudo-metric space. Then for any $f, g \in Q_X$, we have that

$$N(f, g) = \sup\{d(x_1, x_2): x_1, x_2 \in X, d(x_1, x_2) > f_2(x_1), d(x_1, x_2) > g_1(x_2)\}.$$

Proposition 4.5 Let (X, d) be u-injective hull of ultra-quasi-pseudo metric space and let $e_X: X \to Q_X$ be as defined above. Then Q_X is tight extension of $e_X(X)$.

Proof. Let ρ be u-injective hull of ultra-quasi-pseudo metric on Q_X such that $\rho \leq N$ and $\rho(f_x, f_y) = N(f_x, f_y)$ whenever $x, y \in X$. By lemma 3.1 and the fact that $\rho \leq N$, for any

$f, g \in Q_X$, we have

$$N(f, g) = \sup\nolimits_{x_1, x_2 \in X}\{N(f_{x_1}, f_{x_2}): N(f_{x_1}, f_{x_2}) > (f_{x_1}, f), N(g, f_{x_2})\}$$

$$\leq \sup\nolimits_{x_1, x_2 \in X}\{\rho(f_{x_1}, f_{x_2}): \rho(f_{x_1}, f_{x_2}) > \rho(f_{x_1}, f), \rho(g, f_{x_2})\} \leq \rho(f, g),$$

Since $\rho(f_{x_1}, f_{x_2}) \leq \max\{\rho(f_{x_1}, f), \rho(f, g), \rho(g, f_{x_2})\}$. Thus $\rho = N$.

5. MAIN RESULTS

Theorem 5.1 Let (X, d) be u-injective-ultra-quasi-pseudo-metric space. There exists a non-expansive retraction $f: P_X \to Q_X$, that is, a map that satisfies the following conditions:

(a) $R(f(s), f(t), f(u)) \leq R(s, t, u)$ whenever $s, t, u \in P_X$.

(b) $f(s) \leq s$ whenever $s \in P_X$. (In particular we have that $f(s) = s$ whenever $s \in Q_X$.)

Proof. By using Zorn's lemma. Let (X, d) be u-injective-ultra-quasi-pseudo-metric space and let $\wp$ be the set of all maps from P_X to P_X satisfying conditions (a) and (b) in this theorem.

Order $\wp$ by

$$f \leqslant g$$

$$\Leftrightarrow f(s) \leq g(s)$$

$$\Leftrightarrow R(f(s), f(t), f(u)) \leq R(g(s), g(t), g(u)) \tag{5.1.1}$$

For all $s, t, u \in P_X$ and $f, g \in \wp$. Then $\wp \neq \emptyset$ since the identity map belongs to $\wp$.

We have to check now that $\leqslant$ is actually a partial order. Reflexivity is obvious since every map is equal to itself. Let now $f, g, h \in \wp$ such that $f \leqslant g$ and $g \leqslant h$, and $h \leqslant f$. Consider

$$f \leqslant g$$

$$\Rightarrow (f(s))_1 \leq (g(s))_1$$

$$\Rightarrow (f(s))_2 \leq (g(s))_2$$

$$\Rightarrow R(f(s), f(t), f(u)) \leq R(g(s), g(t), g(u))$$

$$g \preccurlyeq h$$

$$\Rightarrow (g(s))_1 \leq (h(s))_1$$

$$\Rightarrow (g(s))_2 \leq (h(s))_2$$

$$\Rightarrow R(g(s), g(t), g(u)) \leq R(h(s), h(t), h(u)).$$

$$h \preccurlyeq f$$

$$\Rightarrow (h(s))_1 \leq (f(s))_1$$

$$\Rightarrow (h(s))_2 \leq (f(s))_2$$

$$\Rightarrow R(h(s), h(t), h(u)) \leq R(f(s), f(t), f(u)). \tag{5.1.2}$$

$(f(s))_1 \leq (g(s))_1 \leq (h(s))_1$ and $(g(s))_1 \leq (h(s))_1 \leq (f(s))_1$ imply that $(f(s))_1 = (g(s))_1 = (h(s))_1$. In a similar manner, we have that $(f(s))_2 = (g(s))_2 = (h(s))_2$ so that we conclude that $f = g = h$.

Also $R(f(s), f(t), f(u)) \leq R(g(s), g(t), g(u))$, $R(g(s), g(t), g(u)) \leq R(h(s), h(t), h(u))$ and $R(h(s), h(t), h(u)) \leq R(f(s), f(t), f(u))$ imply that $f = g = h$. This shows that $\preccurlyeq$ is antisymmetric.

Suppose now that $f, g, h, i \in \wp$ such that $f \preccurlyeq g$, $g \preccurlyeq h$ and $h \preccurlyeq i$,

$$f \preccurlyeq g$$

$$\Rightarrow (f(s))_1 \leq (g(s))_1$$

$$\Rightarrow (f(s))_2 \leq (g(s))_2$$

$$\Rightarrow R(f(s), f(t), f(u)) \leq R(g(s), g(t), g(u))$$

$$g \preccurlyeq h$$

$$\Rightarrow (g(s))_1 \leq (h(s))_1$$

$$\Rightarrow (g(s))_2 \leq (h(s))_2$$

$$\Rightarrow R(g(s), g(t), g(u)) \leq R(h(s), h(t), h(u)).$$

$$h \preccurlyeq i$$

$$\Rightarrow (h(s))_1 \leq (i(s))_1$$

$$\Rightarrow (h(s))_2 \leq (i(s))_2$$

$$\Rightarrow R(h(s), h(t), h(u)) \leq R(i(s), i(t), i(u)). \tag{5.1.3}$$

$(f(s))_1 \leq (g(s))_1 \leq (h(s))_1$ and $(g(s))_1 \leq (h(s))_1 \leq (i(s))_1$ imply that $(f(s))_1 \leq (h(s))_1 \leq (i(s))_1$ by transitivity of $[0, \infty)$ as a subset of $\mathbb{R}$ with the usual ordering $\leq$. Similarly, we can show that

$$(f(s))_2 \leq (h(s))_2 \leq (i(s))_2$$

Also $R(f(s), f(t), f(u)) \leq R(g(s), g(t), g(u))$, $R(g(s), g(t), g(u)) \leq R(h(s), h(t), h(u))$ and $R(h(s), h(t), h(u)) \leq R(i(s), i(t), i(u))$ imply that

$R(f(s), f(t), f(u)) \leq R(h(s), h(t), h(u)) \leq R(i(s), i(t), i(u))$ Thus $f \preccurlyeq i$. This proves that $\preccurlyeq$ is transitive. Therefore

$(\wp, \preccurlyeq)$ is a partially ordered set.

Next we show that every chain in $\wp$ has a lower bound. Let $\emptyset \neq \mathcal{K} \subseteq \wp$ be a chain and defines $h: P_X \to P_X$ by

$$h(s)(x) = \left(\inf_{k \in \mathcal{K}}(k(s))_1(x), \inf_{k \in \mathcal{K}}(k(s))_2(x), \inf_{k \in \mathcal{K}}(k(s))_3(x)\right) \tag{5.1.4}$$

whenever $x \in X$. Since $k(s) \in \wp$, we have that $h(s) \in \wp$. Indeed observe that $h(s) \leq k(s) \leq s, \forall s \in P_X$. Thus $h(s) \leq s$ and condition (a) is satisfied. To check condition (b), $R(h(s), h(t), h(u)) \leq R(k(s), k(t), k(u)) \leq R(s, t, u)$.

Indeed

$$R(s, t, u) \geq R(k(s), k(t), k(u)),$$

Since $k \in \wp$

$$\geq \sup_{x \in X}\left\{r\left((k(s))_1(x), (k(t))_1(x), (k(u))_1(x),\right)\right\}$$

$$\vee \sup_{x \in X}\left\{r\left((k(u))_2(x), (k(t))_2(x), (k(s))_2(x),\right)\right\}$$

$$\geq \sup_{x \in X}\inf_{k \in \mathcal{K}}\left\{r\left((k(s))_1(x), (k(t))_1(x), (k(u))_1(x),\right)\right\}$$

$$\vee \sup_{x \in X}\inf_{k \in \mathcal{K}}\left\{r\left((k(u))_2(x), (k(t))_2(x), (k(s))_2(x),\right)\right\}$$

$$\geq \sup_{x \in X} r\left(\inf_{k \in \mathcal{K}}(k(s))_1(x), \inf_{k \in \mathcal{K}}(k(t))_1(x), \inf_{k \in \mathcal{K}}(k(u))_1(x)\right)$$

$$\vee \sup_{x \in X} r\left(\inf_{k \in \mathcal{K}}(k(u))_2(x), \inf_{k \in \mathcal{K}}(k(t))_2(x), \inf_{k \in \mathcal{K}}(k(s))_2(x)\right)$$

$$= R(h(s), h(t), h(u)). \tag{5.1.5}$$

Thus we have that condition (b) is satisfied and since h is a map from P_X to P_X, we conclude that $h \in \wp$ and h is a lower bound of the chain $\mathcal{K}$ by construction. We therefore appeal to Zorn's lemma to conclude that $\wp$ has a minimal element, say w, with respect to the partial order $\preccurlyeq$. To complete the proof, we show that $w(s) \in Q_X$ whenever $s \in Q_X$.

For each $x \in X$, we have that $p_x \circ w \wp$ and $p_x \circ w \preccurlyeq w$. Thus my minimality of w, we have $p_x \circ w = w$. It therefore follows that, for each $x \in X$, $p_x(w(s)) = w(s)$ whenever $s \in P_X$. Thus the elements in Q_X are defined, we conclude that $w(s) \in Q_X$ whenever $s \in P_X$.

Corollary 5.2. The subspace $[0, \infty)$ of $(\mathbb{R}, u)$ is u-injective.

Conclusion: In this paper we have introduced concepts of $u-$injective hull in ultra-quasi-pseudo metrically tight extension for nonexpansive mappings, also studied q-hyperconvexity in ultra-quasi metric space. We have obtained some interesting result related to nonexpansive retraction in $u-$injective ultra-quasi-pseudo metric space.

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