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Total Coloring of Central Graphs of a Path, a Cycle and a Star

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Abstract: *The total chromatic number of a graph G is defined to be the minimum number of colors needed to color the vertices and edges of a graph in such a way that no two adjacent vertices, no two adjacent edges and no incident vertex and edge are given the same color. In this paper, we discussed the total coloring and total chromatic number of the central graphs of a path, a cycle and a star.*

Keywords: *Central graph; Path; Cycle; Star; Total coloring; Total chromatic number*.

1. Introduction

Bezhad [1] introduced the concept of total coloring and found the chromatic number of some simple graphs. If $G = (V(G), E(G))$ is a graph with the vertex set $V(G)$ and the edge set $E(G)$, a proper total coloring of G is an assignment of colors to the vertices and the edges in such a way that

1. no two adjacent vertices are assigned with the same color,
2. no two adjacent edges are assigned with the same color and
3. no edge and its end vertices are assigned with the same color.

The total chromatic number of a graph G is the minimum number of colors that required to produce a total coloring of and is denoted by $\chi_{tc}(G)$. Bezhad conjectured that for any graph of maximum degree $\Delta(G)$ has a total chromatic number satisfying the condition $\Delta(G) + 1 \leq \chi_{tc}(G) \leq \Delta(G) + 2$. This conjecture is known as the total coloring conjecture (TCC). This conjecture has been verified for many families of graphs and different graphs require different proofs depending on $\Delta(G)$. Bezhad et al. [2] have verified this conjecture for complete graphs and complete multipartite graphs. Rosenfeld [3] proved that the total chromatic number of every cubic graph is totally colorable with five colors. Borodin [4] proved this conjecture for planar graphs. Borodin et al. [5] proved that the chromatic number of a planar graph with maximum degree $\Delta(G) \geq 11$ is $\Delta(G) + 1$. Yap [6] has given the total coloring of r -partite graphs and the graphs with degree $\Delta(G) = 3$, $\Delta(G) = 4$ and $\Delta(G) \geq |G| - 5$. Seoud [7] has discussed about the total coloring of join of two paths, the cartesian product of two paths and the cartesian product of a path and a cycle. Hackmann et.al [8] discussed the circular total coloring of cubic circulant graphs. Sudha et.al [9,10] have proved the total coloring and (k, d)-total coloring of prisms Y_n and the total coloring for a prism graph of n-layers and a grid graph. Chen et al. [11] found that the total chromatic number of generalized Mycielski graphs. Vaidya et al. [12] proved that the total coloring of some cycle related graphs.

Definition 1.1. The central graph of a graph G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G.

The central graph of G is denoted by $C(G)$.

In this paper, we have found the total chromatic number of the central graph of (i) a path, (ii) a cycle and (iii) a star.

2. CENTRAL GRAPH OF A PATH

Consider a path P_5 with the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. If each edge of the path P_5 subdivided exactly once, let the new vertices be denoted by u_1, u_2, u_3 and u_4 as shown in "Fig.1,".

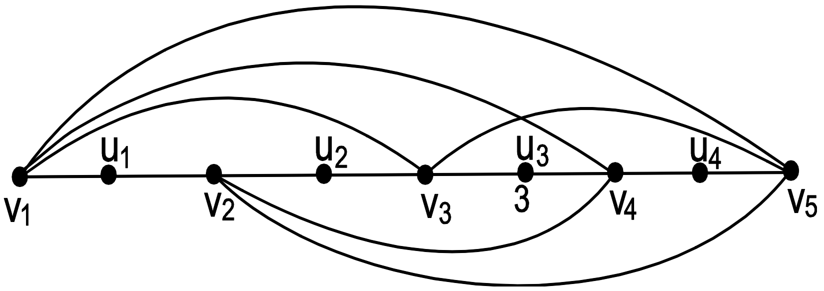

FIGURE 1. *CENTRAL GRAPH OF PATH P_5*

Now join the vertex v_1 with the vertices v_3, v_4 and v_5; the vertex v_2 with the vertices v_4 and v_5 and the vertex v_3 with the vertex v_5. The graph shown in "Fig. 1," is the central graph of the path P_5.

Theorem 2.1. *The total chromatic number of the central graph of a path P_n is given by* $\chi_{tc}(C(P_n)) = \begin{cases} n, & for\ odd\ n \\ n+1, & for\ even\ n \end{cases}$

Proof. Let the path P_n has the vertex set $\{v_i, 1 \le i \le n\}$ and the edge set $\{v_i v_{i+1}, 1 \le i \le n-1\}$. As per the definition of the central graph of the path P_n, let the new vertices be $\{u_i, 1 \le i \le n-1\}$. The vertex set and the edge set of $C(P_n)$ are given by

$$V(C(P_n)) = \{v_i,\ 1 \le i \le n\} \cup \{u_i,\ 1 \le i \le n-1\}$$

$$E(C(P_n)) = \begin{cases} \{v_i u_i, 1 \le i \le n-1\} \cup \{u_i v_{i+1}, 1 \le i \le n-1\} \\ \cup \{v_i v_j, 1 \le i \le n-2,\ i+2 \le j \le n\} \end{cases}$$

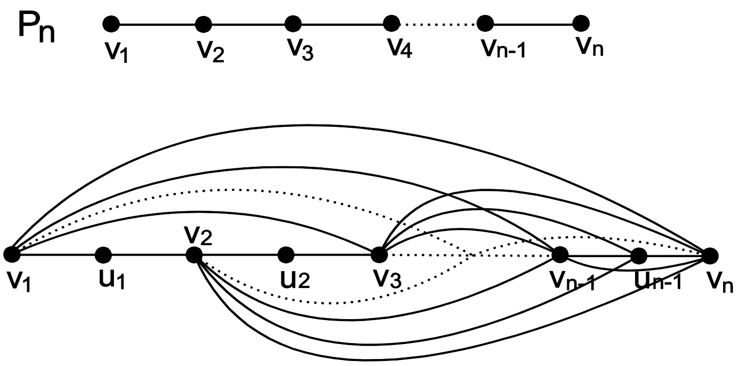

FIGURE 2. *CENTRAL GRAPH OF PATH P_n*

Define the functions f_1 and f_2 from the set of vertices and the set of edges to the set of colors for odd n and even n as follows:

Case 1: Let n be odd.

For all $1 \le i, j \le n$,

$$f_1(v_i) \equiv \begin{cases} (2i+1)(mod\ n), & if\ \ (2i+1) \not\equiv 0(mod\ n) \\ n, & otherwise \end{cases}$$

$$f_1(v_n) = n$$

$$f_1(u_i) \equiv \begin{cases} 2(i+1)(mod\ n), & if\ \ 2(i+1) \not\equiv 0(mod\ n) \\ n, & otherwise \end{cases}$$

$$f_1(u_{n-1}) = 1$$

$$f_2(v_i v_j) \equiv \begin{cases} (i+j)(mod\ n), & if\ \ (i+j) \not\equiv 0(mod\ n),\ j > i+1 \\ n, & otherwise \end{cases}$$

$$f_2(v_iu_j) \equiv \begin{cases}(i+j)(mod\ \ n), & if\ \ (i+j) \not\equiv 0(mod\ \ n)\\ n, & otherwise\end{cases}$$

With this type of coloring, the vertices and the edges of the central graph of the path P_n for odd n are properly total colored with n colors.

Case 2: Let n be even.

For all $1 \le i,j \le n$,

$$f_1(v_i) \equiv \begin{cases}(2i+1)(mod\ \ (n+1)), & if\ \ (2i+1) \not\equiv 0(mod\ \ (n+1))\\ n+1, & otherwise\end{cases}$$

$$f_1(u_i) \equiv \begin{cases}2(i+1)(mod\ \ (n+1)), & if\ \ 2(i+1) \not\equiv 0(mod\ \ (n+1))\\ n+1, & otherwise\end{cases}$$

$$f_2(v_iv_j) \equiv \begin{cases}(i+j)(mod\ \ (n+1)), & if\ \ (i+j) \not\equiv 0(mod\ \ (n+1)),\ \ j > i+1\\ n+1, & otherwise\end{cases}$$

$$f_2(v_iu_j) \equiv \begin{cases}(i+j)(mod\ \ (n+1)), & if\ \ (i+j) \not\equiv 0(mod\ \ (n+1))\\ n+1, & otherwise\end{cases}$$

With this type of coloring, the vertices and the edges of the central graph of the path P_n for even n are properly total colored with $n+1$ colors.

Hence the total chromatic number of the central graph of the path P_n is n for odd n and $n+1$ for even n.

Illustration 2.2. Consider the central graph of a path P_5.

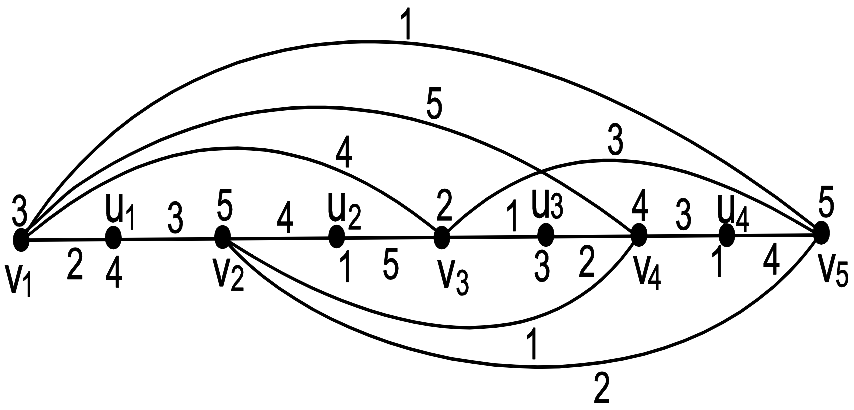

FIGURE 3. *CENTRAL GRAPH OF PATH* P_5

By using the coloring pattern as given in case 1 of theorem 2.1, the colors 1,2,3,4,5 to the vertices and the edges are assigned with the colors as shown in "Fig 3,".

The total chromatic number of the central graph of the path P_5 is 5.

Illustration 2.3. Consider the central graph of a path P_6.

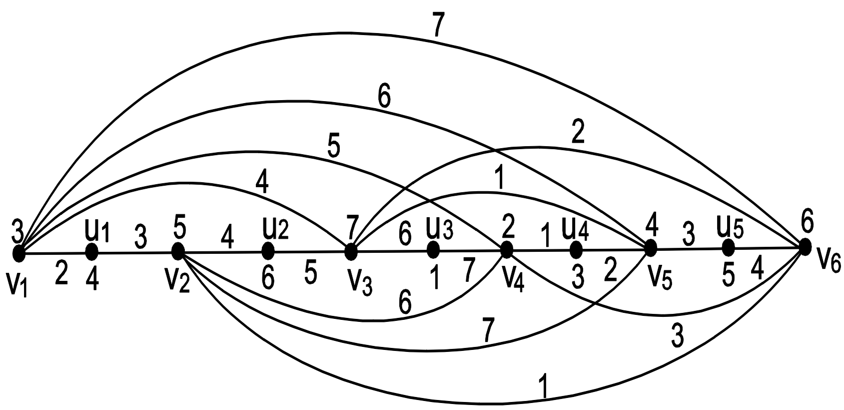

FIGURE 4. *CENTRAL GRAPH OF PATH* P_6

By using the coloring pattern as given in case 2 of theorem 2.1, the colors 1,2,3,4,5,6,7 to the vertices and the edges are assigned with the colors as shown in "Fig 3,".

The total chromatic number of the central graph of the path P_6 is 7.

3. CENTRAL GRAPH OF A CYCLE

Consider a cycle C_5 with the vertex set $\{v_1, v_2, v_3, v_4, v_5\}$. If each edge of the cycle C_5 subdivided exactly once, let the new vertices be denoted by u_1, u_2, u_3, u_4 and u_5 as shown in "Fig. 5,".

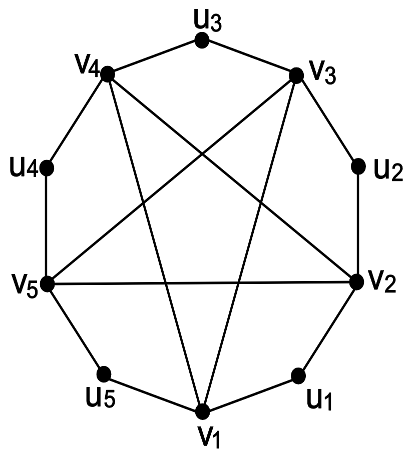

FIGURE 5. *CENTRAL GRAPH OF CYCLE* C_5

Now join the vertex v_1 with the vertices v_3, v_4 and v_5; the vertex v_2 with the vertices v_4 and v_5 and the vertex v_3 with the vertex v_5. The graph shown in "Fig. 5," is the central graph of the cycle C_5.

Theorem 3.1. *The total chromatic number of the central graph of a cycle* C_n *is given by* $\chi_{tc}(C(C_n)) = \begin{cases} n, & for\ odd\ n \\ n+1, & for\ even\ n \end{cases}$

Proof. Let the cycle C_n has the vertex set $\{v_i, 1 \le i \le n\}$ and the edge set $\{v_i v_{i+1}, 1 \le i \le n-1\} \cup \{v_n v_1\}$. As per the definition of the central graph of the cycle C_n, let the new vertices be $\{u_i, 1 \le i \le n\}$. The vertex set and the edge set of $C(C_n)$ are given by

$$V\big(C(C_n)\big) = \{v_i,\ 1 \le i \le n\} \cup \{u_i,\ 1 \le i \le n-1\}$$

$$E\big(C(C_n)\big) = \begin{Bmatrix} \{v_i u_i\ , 1 \le i \le n-1\} \cup \{u_i v_{i+1}, 1 \le i \le n-1\} \\ \cup \{u_n v_1\}\{v_i v_j\ , 1 \le i \le n-2,\ i+2 \le j \le n\} \end{Bmatrix}$$

Define the functions f_1 and f_2 from the set of vertices and the set of edges to the set of colors for odd n and even n as follows:

Case 1: Let n be odd.

For all $1 \le i, j \le n$,

$$f_1(v_i) \equiv \begin{cases} (2i+1)(mod\ n), & if\ \ (2i+1) \not\equiv 0(mod\ n) \\ n, & otherwise \end{cases}$$

$$f_1(u_i) \equiv \begin{cases} 2(i+1)(mod\ n), & if\ \ 2(i+1) \not\equiv 0(mod\ n) \\ n, & otherwise \end{cases}$$

$$f_2(v_i v_j) \equiv \begin{cases} (i+j)(mod\ n), & if\ \ (i+j) \not\equiv 0(mod\ n),\ \ j > i+1 \\ n, & otherwise \end{cases}$$

$$f_2(v_i u_j) \equiv \begin{cases} (i+j)(mod\ n), & if\ \ (i+j) \not\equiv 0(mod\ n) \\ n, & otherwise \end{cases}$$

The above coloring satisfies the condition for a total coloring of the central graph of the cycle C_n for odd n. Hence the total chromatic number of the central graph of the cycle C_n is n for odd n.

Case 2: Let n be even.

For all $1 \le i, j \le n$,

$$f_1(v_i) \equiv \begin{cases} (2i+1)(mod\ \ (n+1)), & if\ \ (2i+1) \not\equiv 0(mod\ \ (n+1)) \\ n+1, & otherwise \end{cases}$$

$$f_1(u_i) \equiv \begin{cases} 2(i+1)(mod\ \ (n+1)), & if\ \ 2(i+1) \not\equiv 0(mod\ \ (n+1)) \\ n+1, & otherwise \end{cases}$$

$$f_1(u_n) = 1$$

$$f_2(v_i v_j) \equiv \begin{cases} (i+j)(mod\ \ (n+1)), & if\ \ (i+j) \not\equiv 0(mod\ \ (n+1)),\ \ j > i+1 \\ n+1, & otherwise \end{cases}$$

$$f_2(v_i u_j) \equiv \begin{cases} (i+j)(mod\ \ (n+1)), & if\ \ (i+j) \not\equiv 0(mod\ \ (n+1)) \\ n+1, & otherwise \end{cases}$$

The above coloring satisfies the condition for a total coloring of the central graph of the cycle C_n for even n. Hence the total chromatic number of the central graph of the cycle C_n is $n+1$ for even n.

Illustration 3.1. Consider the central graph of a cycle C_5.

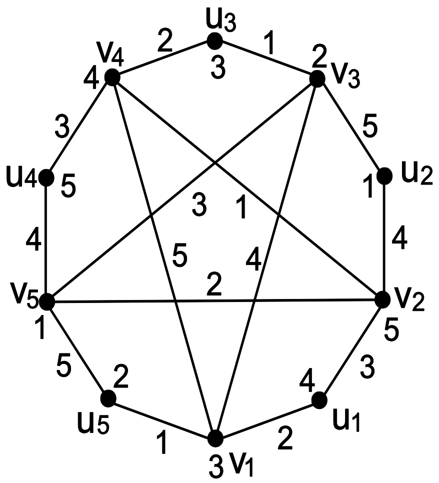

FIGURE 6. *CENTRAL GRAPH OF A CYCLE* C_5

By using the coloring pattern as given in case 1 of theorem 3.1, the colors 1,2,3,4,5 are assigned to the vertices and the edges are colored as shown in "Fig. 6,".

The total chromatic number of the central graph of the cycle C_5 is 5.

Illustration 3.1. Consider the central graph of a cycle C_8.

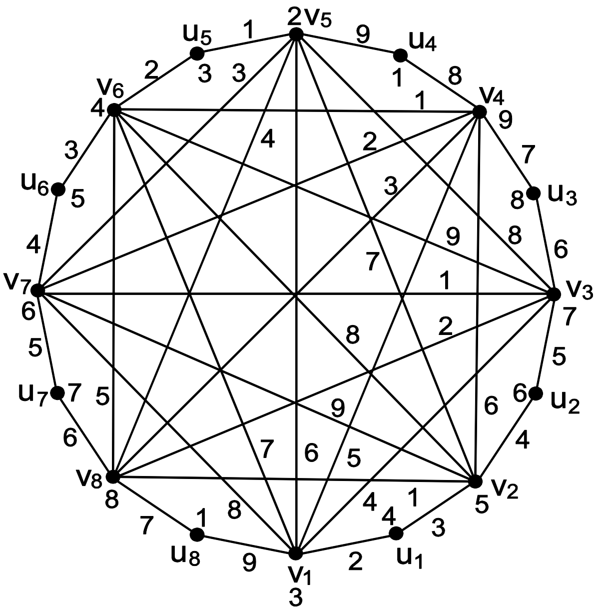

FIGURE 7. *CENTRAL GRAPH OF A CYCLE* C_8

By using the coloring pattern as given in case 2 of theorem 3.1, the colors 1,2,3,4,5,6,7,8,9 are assigned to the vertices and the edges are colored as shown in "Fig. 7,".

The total chromatic number of the central graph of the cycle C_8 is 9.

4. CENTRAL GRAPH OF A STAR

Consider a star $K_{1,4}$ with the vertex set $\{v_0, v_1, v_2, v_3, v_4\}$. If each edge of the star $K_{1,4}$ is supersubdivided by the biparitite graph $K_{2,1}$, let the new vertices be denoted by u_1, u_2, u_3 and u_4 as shown in "Fig. 8,".

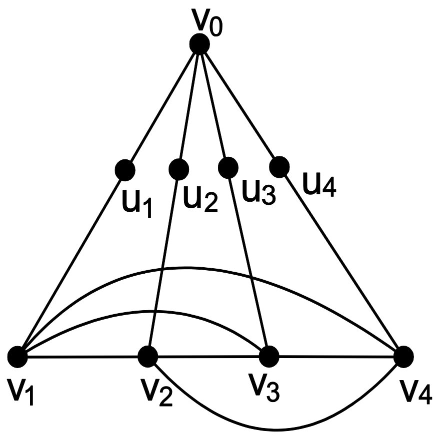

FIGURE 8. *CENTRAL GRAPH OF A STAR* $K_{1,4}$

No join the vertex v_1 with the vertices v_2, v_3 and v_4; the vertex v_2 with the vertices v_3 and v_4 and the vertex v_3 with the vertex v_4. The graph shown in "Fig. 8," is the central graph of the star $K_{1,4}$.

Theorem 4.1. The total chromatic number of the central graph of a star $K_{1,n}$ is $n + 1$ for $n \geq 3$.

Proof.

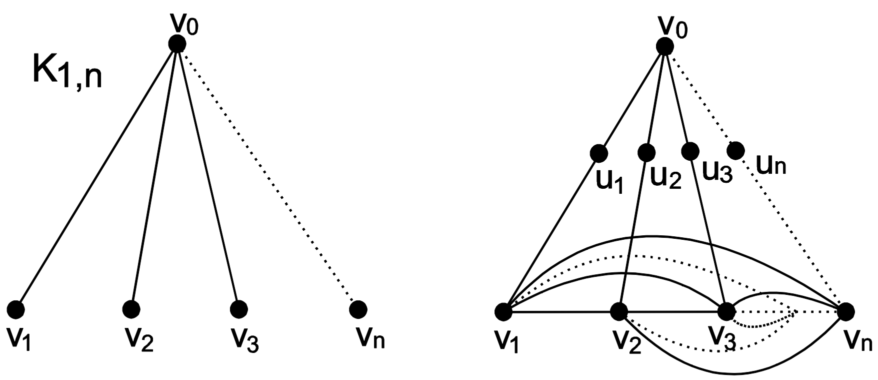

FIGURE 9. *CENTRAL GRAPH OF A STAR* $K_{1,n}$

Let v_0 be the central vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendent vertices of $K_{1,4}$. As per the definition of the central graph, let the new vertices be $u_i, 1 \leq i \leq n$.The vertex set and the edge set of $C(K_{1,n})$ are given by

$$V\left(C(K_{1,n})\right) = \{v_i, 1 \leq i \leq n\} \cup \{u_i, 1 \leq i \leq n\} \cup \{v_0\}$$

and $$E\left(C(K_{1,n})\right) = \begin{cases} \{v_0 u_i, 1 \leq i \leq n\} \cup \{u_i v_i, 1 \leq i \leq n\} \\ \cup \{v_i v_j, 1 \leq i \leq n-1, i+1 \leq j \leq n\} \end{cases}$$

Define the function f_1 and f_2 from the set of vertices and the set of edges to the set of colors $\{1,2,3, \ldots\ldots, n\}$ as follows:

$$f_1(v_0) = n + 1,$$

$$f_1(v_i) = i, 1 \leq i \leq n ,$$

$$f_1(u_1) = 3 \text{ and } f_1(u_n) = 1$$

$$f_1(u_i) = i - 1,\ 2 \leq i \leq n\text{-}1,$$

$$f_2(v_0 u_i) = i, 1 \leq i \leq n,$$

$$f_2(v_i u_i) = \begin{cases} 2i\ (mod(n+1)),\ if\ 2i \not\equiv 0(mod(n+1)), \\ n+1\ , otherwise \end{cases}$$

$$f_2(v_i u_i) = \begin{cases} (i+j)\ (mod(n+1)),\ if\ (i+j) \not\equiv 0(mod(n+1)), j \geq i \\ n+1\ , otherwise \end{cases}$$

$$for\ 1 \leq i, j \leq n.$$

The above coloring satisfies the condition for a total coloring of the central graph of the star $K_{1,n}$.

Hence the total chromatic number of the central graph of the star $K_{1,n}$ is $n + 1$ for $n \geq 3$.

Illustration 4.1. Consider the central graph of a star $K_{1,6}$.

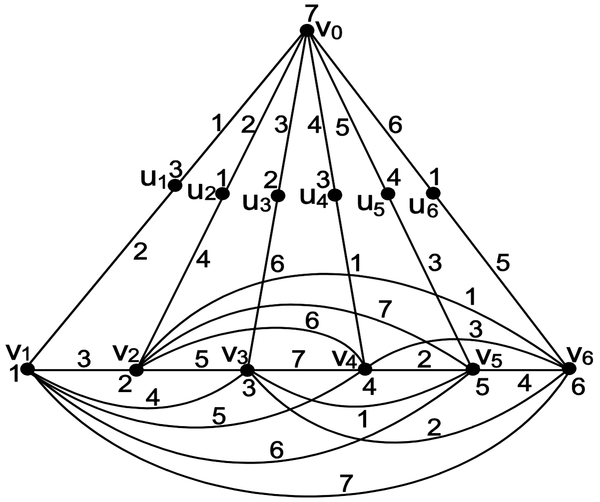

FIGURE 10. *CENTRAL GRAPH OF A STAR* $K_{1,6}$

By using the coloring pattern given in theorem 4.1, the colors 1,2,3,4,5,6,7 are assigned to the vertices and the edges with the colors as shown in "Fig. 10,"

5. Conclusion

The total coloring of the central graph of a path, a cycle and star are discussed in this paper and found the total chromatic numbers to be

1. $\chi_{tc}(C(P_n)) = \begin{cases} n, & for\ odd\ n \\ n+1, & for\ even\ n \end{cases}$
2. $\chi_{tc}(C(C_n)) = \begin{cases} n, & for\ odd\ n \\ n+1, & for\ even\ n \end{cases}$
3. $\chi_{tc}(C(K_{1,n})) = n + 1$.

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