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Total Dominating Color Transversal number of Product of Graphs

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Abstract: *Total Dominating Color Transversal Set is the combination of three concepts of graph theory, viz., Total Dominating Set, Transversal and Proper Coloring of vertices of a graph. It is defined as a Total Dominating Set which is also transversal of some χ - Partition of vertices of G. Here χ is the Chromatic number of the graph G. Total Dominating Color Transversal number of a graph is the cardinality of a Total Dominating Color Transversal Set which has minimum cardinality among all such sets that the graph admits. In this paper, we consider two graph products namely; Cartesian product and Kronecker product. We determine Total Dominating Color Transversal number of Cartesian product of Complete graphs and of Kronecker product of Complete k - Partite graphs. We find a necessary and sufficient condition under which this number for Cartesian product attains its lower bound and we also obtain lower bound of this number for Kronecker product. Additionally, we provide other related results with sufficient number of examples, wherever required.*

Keywords: *Total Dominating Color Transversal Set; χ – Partition of a graph; Cartesian product; Kronecker product.*

1. Introduction

We begin with simple, finite, connected and undirected graph without isolated vertices. We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields χ equivalence classes. Transversal of a χ - Partition of G is a collection of vertices of G that meets all the color classes of the χ – Partition. That is, if T is a subset of V(the vertex set of G) and $\{V_1, V_2, ..., V_\chi\}$ is a χ - Partition of G then T is called a Transversal of this χ - Partition if $T \cap Vi \neq \emptyset$, $\forall\, i \in \{1, 2,...., \chi\}$.Total Dominating Color Transversal Set of graph G is a Total Dominating Set with the extra property that it is also Transversal of some such χ - Partition of G.

We first mention definitions.

2. Definitions

Definition 2.1[4]: (Total Dominating Set) Let G = (V, E) be a graph. Then a subset S of V (the vertex set of G) is said to be a Total Dominating Set of G if for each $v \in V$, v is adjacent to some vertex in S.

Definition 2.2[4]: (Minimum Total Dominating Set/Total Domination number) Let G = (V, E) be a graph. Then a Total Dominating set S is said to be a Minimum Total Dominating set of G if $|S|$ = minimum $\{|D|$: D is a Total Dominating set of G$\}$. Here S is called γ_t −set and its cardinality, denoted by $\gamma_t(G)$ or just by γ_t, is called the Total Domination number of G.

Definition 2.3[1]: (χ -partition of a graph) Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ - Partition of the graph G.

Definition 2.4[1]: (Transversal of a χ - Partition of a graph) Let G = (V, E) be a graph with χ – Partition $\{V_1, V_2, ..., V_\chi\}$. Then a set S ⊂ V is called a Transversal of this χ – Partition if $S \cap V_i \neq \emptyset$, $\forall$ i ∈ {1, 2, 3, ..., χ}.

Definition 2.5[1]: (Total Dominating Color Transversal Set) Let G = (V, E) be a graph. Then a Total Dominating Set S ⊂ V is called a Total Dominating Color Transversal Set of G if it is Transversal of at least one χ - Partition of G.

Definition 2.6[1]: (Minimum Total Dominating Color Transversal Set/ Total Dominating Color Transversal number) Let G = (V, E) be a graph and S ⊂ V be a Total Dominating Transversal Set of G. Then S is said to be a Minimum Total Dominating Color Transversal Set of G if |S| = minimum {|D|: D is a Total Dominating Color Transversal Set of G}. Here S is called γ_{tstd} −set and its cardinality, denoted by γ_{tstd} (G) or just by γ_{tstd}, is called the Total Dominating Color Transversal number of G.

Definition 2.7 [13]: (Cartesian product of Graphs)

The Cartesian product G □ H of graphs G and H is the graph with vertex set V (G)×V (H) and edge set {{(u, x), (v, y)} | either u = v and x is adjacent to y in H or u is adjacent to v in G and x = y}.

Definition 2.8 [13]: (Kronecker / Direct/ Tensor/ product of Graphs)

The Kronecker product G×H of graphs G and H is the graph with vertex set V (G)×V (H) and edge set {{(u, x), (v, y)} | {u, v} ∈ E(G) and {x, y} ∈ E(H)}.

3. Main Results

First we state the following theorem taken from [1].

Theorem 3.1 [1]: If G is a graph with χ (G) = 2 then γ_{tstd}(G) = γ_t(G).

We first discuss about Cartesian product of graphs. Below given remark will prove useful.

Remark 3.2:

1) By [9] we know that χ (G □ H) = max {χ (G), χ (H)}.

2) G □ H ≅ H □ G. So γ_{tstd} (G □ H) = γ_{tstd} (H □ G).

3) This operation is commutative if the graphs are not labeled.

4) G □ H is connected if and only if both G and H are connected.

Theorem 3.3: Let G and H be two graphs. γ_{tstd} (G□H) = 2 if and only if

(1) χ (G) = χ (H) = 2 and

(2) At least one of G or H is a path graph with two vertices and γ(G) = γ(H) = 1.

Proof: Note that both G and H are connected graphs with δ (G) ≥ 1 and δ (H) ≥ 1.

Suppose γ_{tstd} (G□H) = 2. Then χ (G□H) ≤ 2 implies that χ (G□H) = 2. As χ (G□H) = max{ χ (G), χ (H)} = 2, χ (G) = χ (H) = 2, which proves (1).

Also γ_{tstd} (G□H) = 2 implies that γ_t (G□H) = 2. So there exists a γ_t - Set D = {(a, x), (b, y)} of G□H which is also γ_{tstd} − Set of G□H. Obviously either a = b or x = y.

Assume a= b. Then x is adjacent to y in H. Therefore D = {(a, x), (a, y)}.

Let c ≠ a be other vertex of G.

Claim: V (H) = {x, y}

Suppose |V (H)| > 2. Then there exists z ∈ V (H) \ {x, y}. Then clearly (c, z) cannot be dominated by any vertex in D = {(a, x), (a, y)}, which is contradiction as D is a Total Dominating Set of G□H. Hence V (H) = {x, y}. Therefore H is a path with two vertices.

claim: {a} is a Dominating Set of G.

If {a} is not a dominating set of G then a is not adjacent to some vertex d($\neq$ a) of G. Then (d, x) cannot be dominated by any vertex in D = {(a, x), (a, y)}, which is contradiction as D is a Total Dominating Set of G□H. Therefore {a} is a Dominating Set of G.

Hence $\gamma(G) = \gamma(H) = 1$.

Assuming x = y then, similarly as above, we prove that G is a path graph with two vertices and {x} is a dominating set of H. Hence again we prove at least one of G or H is a path graph with two vertices and $\gamma(G) = \gamma(H) = 1$.

Conversely assume both (1) and (2). Assume without loss of generality that H is a path graph with two vertices x and y. Assume that {a} is a Dominating Set of G.

Consider D = {(a, x), (a, y)}. Let b be any other vertex of G. Then (b, z) (where z = x or z = y) is adjacent to some vertex of D. Also (a, x) and (a, y) are adjacent. So D is Total Dominating Set of G□H. Hence γ_t(G□H) = 2.

As χ (G) = χ (H) = 2, χ (G□H) = 2. By theorem 1, γ_{tstd} (G□H) = γ_t(G□H) = 2.

Remark 3.4: Phrase 'At least' appears in the above theorem 3. 3 because the theorem holds even if both G and H are path graphs with two vertices. Let us see, by example, justification of this statement.

Example 3.5:

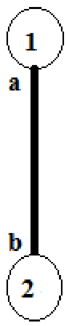

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Fig. 1

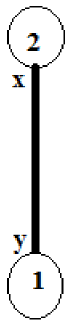

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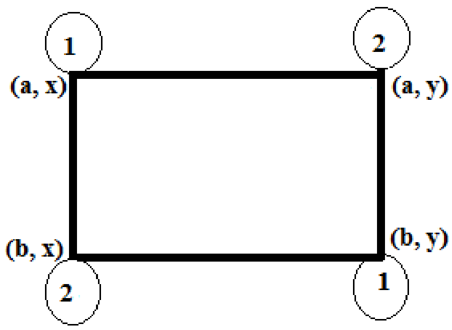
Fig. 2



G□H

Fig. 3

Clearly γ_{tstd} (G□H) = 2. Both G and H are path graphs with two vertices and $\gamma(G) = \gamma(H) = 1$.

Remark 3.6: We know that if the Domination number of a graph is one then the Total Domination number of the graph is two. On this basis we state the following corollary to our theorem 3.3

Corollary 3.7: Let G and H be two graphs. If γ_{tstd} (G□H) = 2 then

(1) χ (G) = χ (H) = 2

(2) $\gamma_t(G) = \gamma_t(H) = 2$.

Remark 3.8: Converse of above corollary 3.7 is not true, in general. The below given example justifies it.

Example 3.9: Consider two graphs given below.

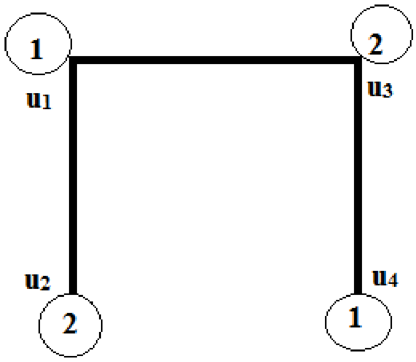


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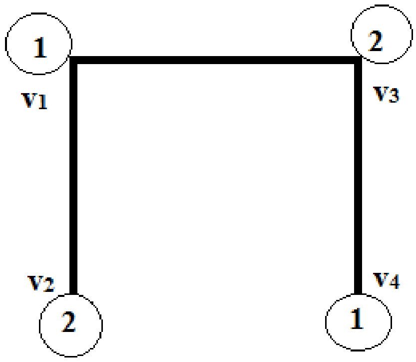
Fig. 4



H

Fig. 5

Clearly χ (G) = χ (H) = 2 and $\gamma_t(G) = \gamma_t(H) = 2$.

Assume γ_{tstd} (G□H) = 2. Then by theorem 3.3, γ (G) = 1 which is contradiction as γ (G) = 2. So γ_{tstd} (G□H) > 2.

Theorem 3.10: $\boldsymbol{\gamma_{tstd}}$ **(K_m□K_n) = max {m, n}**

Proof: We know that χ (K_m □ K_n**)** = max {m, n}. Assume without loss of generality that max {m, n} = n. So γ_{tstd} (K_m □ K_n) $\geq \chi$ **(**K_m □ K_n**)** = n. Note that K_m □ K_n have mn vertices. Consider the following table of mn vertices where u_i's and v_j's, respectively, indicate the vertices of K_m and K_n.

Table 1

(u_1, v_1)	(u_1, v_2)	(u_1, v_3)	....	(u_1, v_n)
(u_2, v_1)	(u_2, v_2)	(u_2, v_3)	....	(u_2, v_n)
(u_3, v_1)	(u_3, v_2)	(u_3, v_3)	....	(u_3, v_n)
....	....	....	....	....
(u_m, v_1)	(u_m, v_2)	(u_m, v_3)	....	(u_m, v_n)

For any fix i, $1 \leq i \leq m$, $S_i = \{(u_i, v_j) : 1 \leq j \leq n\}$ is a Total Dominating Set of K_m □ K_n. Note that <S_i > (for every i, $1 \leq i \leq m$) is a clique of K_m□K_n of order n. Hence assigning n distinct colors to all the vertices in S_i, S_i becomes a transversal of some χ – Partition of K_m□K_n . Therefore γ_{tstd} (K_m□K_n) = n = max {m, n}.

Now we discuss about Kronecker product of graphs. We begin with remark.

Remark 3.11:

1) Kronecker Product is commutative and associative.
2) [14] G × H is connected if and only if both graphs are connected and at least one graph is non bipartite.
3) By [8], G × H is bipartite if and only if at least one of G and H is bipartite.
4) [13] χ(G × H) $\leq$ min { χ (G), χ (H)}.

Theorem 3.12[6]. For any two graphs G and H, we have $\boldsymbol{\gamma_t}$ **(G × H)≤** $\boldsymbol{\gamma_t}$ **(G)** $\boldsymbol{\gamma_t}$ **(H).**

Lemma 3.13[7] If G = (V_0 ∪ V_1,E) and H = (W_0∪ W_1, F) are bipartite graphs, then (V_0 ×W_0) ∪ (V_1 ×W_1) and (V_0 ×W_1) ∪ (V_1 ×W_0) are vertex sets of the two components of G × H.

Theorem 3.14: Let G and H be two Graphs. Then $\boldsymbol{\gamma_{tstd}}$ **(G × H) ≥ 3.**

Proof: Let γ_{tstd} (G × H) = 2. Then γ_t (G × H) = 2. So there exists a γ_t - set D = {(a, b), (c, d)} (say) of G × H. But vertex (a, d) cannot be dominated by any vertex in D. Hence we get a contradiction.

Hence the theorem.

Remark 3.15: The lower bound in the above theorem 3.14 is sharp. Example 3.18 justifies this.

Theorem 3.16: If G and H are, respectively, complete K_1 and K_2 Partite graphs

then $$\gamma_{tstd}(G \times H) = \begin{cases} 4, & K_1 = 2 \text{ or } K_2 = 2 \\ \chi(G \times H), & \text{Otherwise.} \end{cases}$$

Proof: Case 1: K_1= 2 and K_2 = 2

As χ (G × H) = 2 by theorem 1, γ_{tstd} (G × H) = γ_t (G × H).

G × H is disconnected with two components and each component has at least two vertices. So γ_{tstd} (G × H) ≥ 4. Also So by theorem 3.12, γ_{tstd} (G × H) ≤ γ_t (G) γ_t (H) = 2 . 2 = 4. Theorefore γ_{tstd} (G × H) = 4.

Case 2: Either K_1= 2 or K_2 = 2.

Note that in this case G × H is connected.

As Kronecker Product is Commutative , we assume without loss of generality K_1= 2 and $K_2 > 2$.

So χ (G × H) = 2. Hence by theorem 1, γ_{tstd} (G × H) = γ_t (G × H).

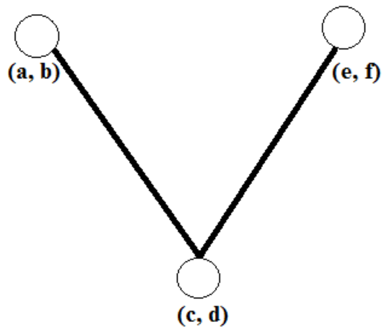
Claim : Every Total Dominating Set of G × H contains four vertices.

Suppose that D is Total Dominating Set of G × H.

Assume |D| = 3.

Let D = {(a, b), (c, d), (e, f)} . Note that < D> is a connected graph as we are dealing with Total Domination theory.

As χ (G × H) = 2, <D> is not complete sub graph of G × H. So assume without loss of generality that (a, b) is adjacent to (c, d) and (c, d) is adjacent to (e, f).



<D> Fig. 6

Here we first note that as G and H are, respectively, Complete K_1- Partite and Complete K_2 - Partite graphs they have unique χ – Partitions.

(a, b) is adjacent to (c, d) implies that a is adjacent to c in G. Hence color class of a and color class of c are different. (c, d) is adjacent to (e, f) implies that c is adjacent to e in G. Hence color class of c and color class of e are different. Therefore a and e are in same color class , as G is bipartite graph. Hence a and e are not adjacent.

Definitely (a, d) ∈ V (G×H) cannot be dominated by (a, b) and (c, d). Also also as a and e are not adjacent (a, d) cannot be dominated by (e, f) as well. So D is not a Dominating Set, which is contradiction to our assumption that D is a Total Dominating Set of G × H. Hence $|D| \geq 4$.

Therefore γ_t (G × H) ≥ 4 and so γ_{tstd} (G × H)) ≥ 4. Also by theorem 3.12, γ_{tstd} (G × H) ≤ γ_t (G) γ_t (H) = 2 . 2 = 4. Therefore γ_{tstd} (G × H) = 4.

Case 3: $K_1 > 2$ and $K_2 > 2$.

The graph G × H is connected.

In this case, χ (G × H) > 2.

We know that γ_t (G × H) ≥ 3, by theorem 3.14.

Consider D = {(a, b), (c, d), (e, f)} ⊂V(G × H) such that a, c and e are in different color classes of the K_1 - Partition of G and b, d and f are in different color classes of the K_2 - Partition of H. Then <D> is complete sub graph of G × H.

Claim: D is a Total Dominating Set of G × H.

Consider (u, v) ∈ V(G × H). If (u, v) ∈ D then as <D> is complete is a complete graph, it is dominated by all the other vertices of D. So assume (u, v) ∈ V(G × H) \ D. Suppose (u, v) is not dominated by (a, b) then u is not adjacent to a in G or v is not adjacent to b in H.

Sub Case 1: u is not adjacent to a in G and v is not adjacent to b in H.

Then color class of u and a is same in G and color class of v and b is same in H. Then as (a, b) is adjacent to (c, d) the color classes of a and c are different in G and color classes of b and d are different in H. So u is adjacent to c in G and v is adjacent to d in H. Hence (c, d) dominates (u, v).

Sub Case 2: Either u is not adjacent to a in G or v is not adjacent to b in H.

Without loss of generality assume that u is not adjacent to a in G and v is adjacent to b in H. So color class of u and a is same in G and v and b are in different color classes in H. Trivially as a, c and e are in different color classes in G, u is adjacent to c and e both. Now if v is adjacent to d then (c, d) dominates (u, v) and if v is not adjacent to d then v is adjacent to f as color class of v and d is same and hence (e, f) dominates (u, v).

Hence from both the above two cases D is a Minimum Total Dominating Set of G × H with <D> a complete subgraph of G × H. Assign distinct colors to distinct vertices of <D>. If K = χ (G × H). Add one vertex from each remaining K – 3 color classes of the K – Partition of G × H. The resultant set becomes a Minimum Total Dominating Color Transversal of G× H with cardinality K.

Hence the theorem.

Corollary 3.17: $\gamma_{tstd}(K_m \times K_n) = \begin{cases} 4, & m = 2 \text{ or } n = 2. \\ \min\{m, n\}, & \text{otherwise}. \end{cases}$

Proof: By theorem 3.16 and as $\chi(K_m \times K_n) = \min\{m, n\}$.

Example 3.18: Consider two graphs G and H as shown below:

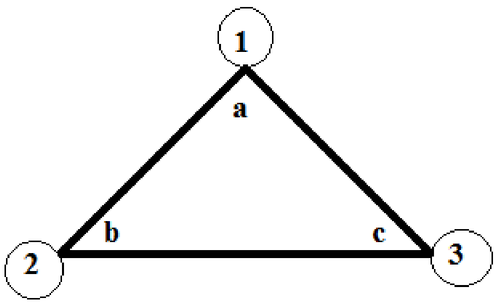


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Fig. 7

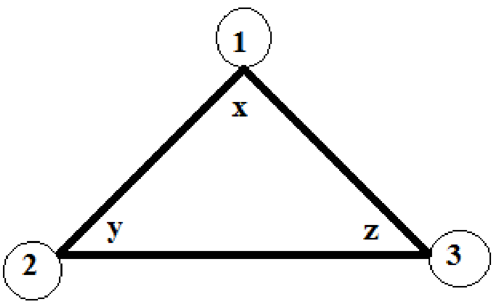

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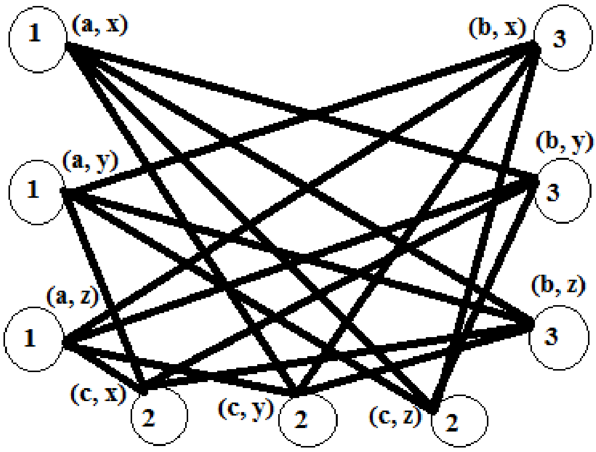
Fig. 8



G × H

Fig. 9

Cleary {(a, x), (b, y), (c, z)} is a γ_{tstd} - Set of G. Hence γ_{tstd} (G) = 3 = χ (G × H), which justifies the theorem 3.16 and corollary 3.17.

4. Conclusion

We have explored some properties of Total Dominating Color number of two types of products of graphs viz; Cartesian product and Kronecker product. Discussion of this number for other graph products like Lexicographic product, Strong product is also possible. One may discuss about this number for product of cycle graphs, path graphs and other graphs. One this is for sure that for any further discussion of this number for different product of graphs, in general, will require rigorous work and analysis on known results of Proper coloring and Total Domination in graphs. One may have to explore some more preliminaries for further discussion.

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