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Changing and unchanging of Total Dominating Color Transversal number of Graphs

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Abstract: *Total Dominating Color Transversal Set of a graph is a Total Dominating Set of the graph which is also Transversal of Some χ - Partition of vertices of the graph. Here χ is the Chromatic number of the graph. Total Dominating Color Transversal number of a graph is the cardinality of a Total Dominating Color Transversal Set which has minimum cardinality among all such sets that the graph admits. In this paper, we determine the necessary and sufficient conditions under which this number decreases, increases or remains same when a vertex is removed from a graph. We provide sufficient number of examples to justify our results.*

Keywords: *Total Dominating Color Transversal number, χ – Partition of a graph and Transversal.*

1. Introduction

We begin with simple, finite, connected and undirected graph without isolated vertices. We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields χ equivalence classes. Transversal of a χ - Partition of G is a collection of vertices of G that meets all the color classes of the χ – Partition. That is, if T is a subset of V(the vertex set of G) and $\{V_1, V_2, ..., V_\chi\}$ is a χ - Partition of G then T is called a Transversal of this χ - Partition if $T \cap Vi \neq \emptyset$, $\forall$ i $\in$ {1, 2,...., χ }.Total Dominating Color Transversal Set of graph G is a Total Dominating Set with the extra property that it is also Transversal of some such χ - Partition of G. Total Dominating Color Transversal Set of G with minimum cardinality, among all such sets that the graph G admits, is called Minimum Total Dominating Color Transversal Set of G and its cardinality, denoted by γ_{tstd}(G) or just by γ_{tstd}, is called the Total Dominating Color Transversal number of G.

In this paper, we compute the operation of removing a vertex and analyze the effect of this operation on γ_{tstd}. As we are dealing with Total Domination theory, we remove only those vertices whose removal does not yield isolated vertices in the resultant sub graph.

First let us go through some definitions.

2. Definitions

Definition 2.1[4]: (Total Dominating Set) Let G = (V, E) be a graph. Then a subset S of V (the vertex set of G) is said to be a Total Dominating Set of G if for each v $\in$ V, v is adjacent to some vertex in S.

Definition 2.2[4]: (Minimum Total Dominating Set/Total Domination number) Let G = (V, E) be a graph. Then a Total Dominating Set S is said to be a minimum Total Dominating Set of G if |S| = minimum {|D|: D is a Total Dominating Set of G}. Here S is called γ_t −set and its cardinality, denoted by γ_t(G) or just by γ_t, is called the Total Domination number of G.

Definition 2.3[1]: (χ -partition of a graph) Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ - Partition of the graph G.

Definition 2.4[1]: (Transversal of a χ - Partition of a graph) Let G = (V, E) be a graph with χ – Partition {V_1, V_2, ..., V_χ}. Then a set S ⊂ V is called a Transversal of this χ – Partition if S ∩ V_i≠ ∅, ∀ i ∈ {1, 2, 3, ..., χ}.

Definition 2.5[1]: (Total Dominating Color Transversal Set) Let G = (V, E) be a graph. Then a Total Dominating Set S ⊂ V is called a Total Dominating Color Transversal Set of G if it is Transversal of at least one χ – Partition of G.

Definition 2.6[1]: (Minimum Total Dominating Color Transversal Set/Total Dominating Color Transversal number) Let G = (V, E) be a graph. Then S ⊂ V is called a Minimum Total Dominating Color Transversal Set of G if |S| = minimum {|D|: D is a Total Dominating Color Transversal Set of G}. Here S is called γ_{tstd} −Set and its cardinality, denoted by γ_{tstd}(G) or just by γ_{tstd}, is called the Total Dominating Color Transversal number of G.

Definition 2.7: (Isolated Vertex) Let G = (V, E) be a graph. Then a vertex v of G is isolated vertex if its degree is 0.

Definition 2.8: (Isolate vertex of a set) Let G = (V, E) be a graph and S be a subset of V. Then a vertex v in S is called isolate of S if v is not adjacent to any vertex in S.

Definition 2.9: (Open and Closed neighbourhood of a vertex) Let G = (V, E) be a graph. Then Open neighbourhood of a vertex v of G is denoted and defined as N (v) = {u ∈ V / u is adjacent to v} and Closed neighbourhood of v is denoted and defined as N [v] = N (v) ∪ {v}.

Definition 2.10: (Total Chromatic neighbourhood of a vertex) Let G = (V, E) be a graph. Let S be a Total Dominating Color Transversal Set of G for some χ – Partition Π = {V_1, V_2,...., V_χ} of G and v ∈ S with v ∈ V_i, for some V_i ∈ Π . Then the Total Chromatic neighbourhood of v with respect to S and Π is denoted and defined as χ^T(v, S, Π) = {u∈ S/ u ∈ V_i and S \ {u} does not have isolate vertex}.

We here note that v may not belong to χ^T(v, S, Π).

3. MAIN RESULTS

Result 3.1 [7]: Let G be a graph and H be a sub graph of G. Then χ (H) ≤ χ (G).

Result 3.2: Let G= (V, E) be a graph and v ∈ V. If χ (G \ {v}) < χ (G) then χ (G \ {v}) + 1 = χ (G).

Theorem 3.3[9] : Let G = (V, E) be a graph and v ∈ V. Then χ (G \ {v}) = χ (G) if and only if {v} is not a color class of any χ - Partition of G.

Proof: Assume χ (G \ {v}) = χ (G). Suppose {v} is a color class of some χ – Partition ∏ = {V_1, V_2,...., V_χ} of G with {v}= V_1. Clearly { V_2,...., V_χ} is a χ – 1 partition of G \ {v}, which is contradiction. So {v} is not a color class of any χ - Partition of G.

Conversely assume that {v} is not a color class of any χ - Partition of G. Suppose χ (G \ {v}) < χ (G). Let {V_1, V_2,...., $V_{\chi-1}$}be a χ - 1 Partition of G \ {v}. Then {{v}, V_1, V_2,...., $V_{\chi-1}$} is a χ – Partition of G, which is contradiction. Hence χ (G \ {v}) = χ (G).

Before going further we introduce some notations.

Notations: Let G = (V, E) be a graph and G \ {v} be a sub graph of G obtained by removing vertex v from G.

(1) V^i = {v ∈ V/ G \ {v} has an isolated vertex}

(2) V^0_{tstd} = {v ∈ V/ Υ_{tstd} (G \ {v}) = Υ_{tstd} (G)}

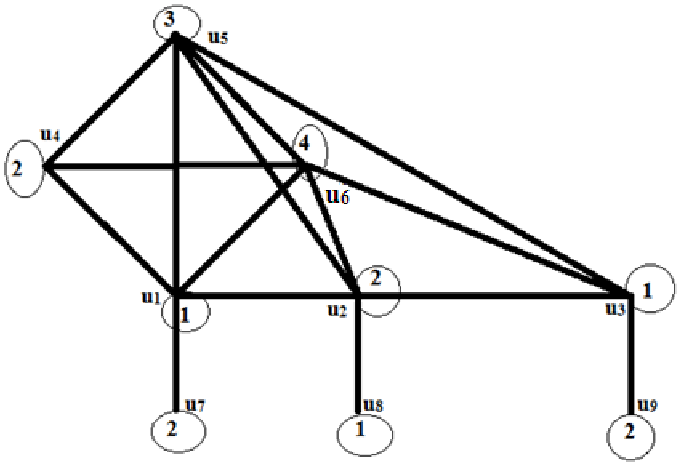
(3) V^-_{tstd} = {v ∈ V/ Υ_{tstd} (G \ {v}) < Υ_{tstd} (G)}

(4) V^+_{tstd} = {v ∈ V/ Υ_{tstd} (G \ {v}) > Υ_{tstd} (G)}

Clearly all the above sets are mutually disjoint and their union is the vertex set V of G.

We first of all provide examples that reflects the fact that Υ_{tstd} number may decrease, increase or remain same by removing a vertex.

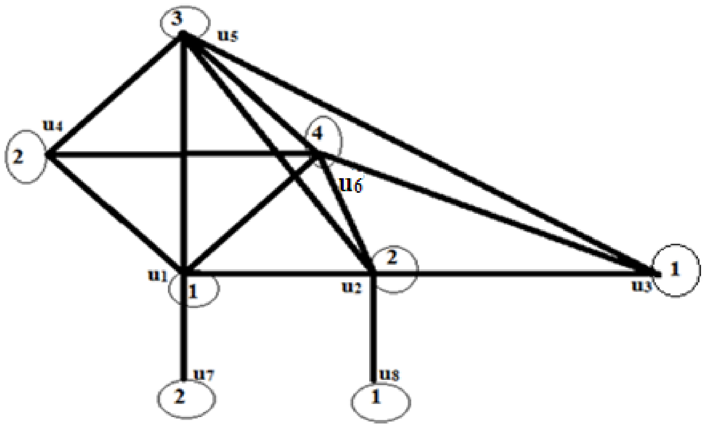
Example 3.4: For the given graph G , in Fig. 1, Υ_{tstd} number decreases by removing vertex u_9.



G

Fig. 1

Υ_{tstd} (G) = 5

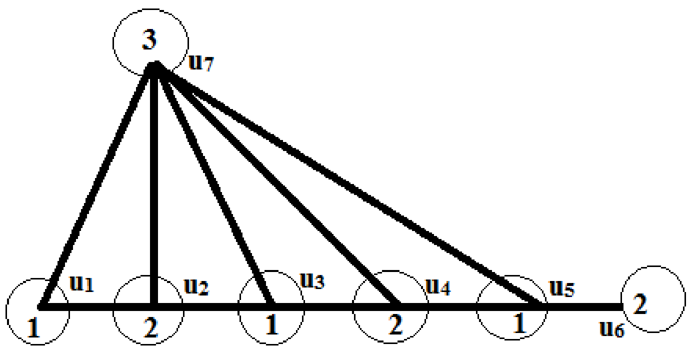


G \ {u_9}

Fig. 2

Υ_{tstd} (G \ {u_9}) = 4

Example 3.5: For the given graph G, in Fig. 3, Υ_{tstd} number increases by removing a vertex u_7.



G

Fig. 3

Υ_{tstd} (G) = 3 and Υ_{tstd} (G \ {u_7}) = Υ_{tstd} (P_6) = 4 (Here P_6 is a path graph with 6 vertices)

Example 3.6: For the graph G = P_4 (a path graph with four vertices). We know that Υ_{tstd} (G) = 2. Note that if v is a pendant vertex of G then G \ {v} is a path graph with three vertices and Υ_{tstd} (G \ {v}) = 2. Hence Υ_{tstd} (G) = Υ_{tstd} (G \ {v}).

Theorem 3.7: Let G = (V, E) be a graph and v ∈ V with v ∉ V^i , {v} is not a color class of any χ - Partition of G and deg (v) < χ. Then v ∈ V^-_{tstd} if and only if following two conditions are satisfied:

(1) There exists a Υ_{tstd} - Set S of G not containing v.

(2) There exists u∈ N (v) such that u ∈ χ^T(w, S, Π), for some vertex w (≠ u) and χ - Partition Π of G for which S is a transversal.

Proof: As {v} is not a color class of any χ - Partition of G, χ (G \ {v}) = χ (G). (by theorem 3.3)

Assume v ∈ V^-_{tstd} .

Let D be a Υ_{tstd} - Set of G \ {v} for some χ – Partition Π' = { V_1, V_2,...., V_χ} of G \ {v}. As deg (v) < χ , {v} cannot meet all the color classes of G \ {v}. So there exists a color class, say Vi, of G \ {v} such {v} ∩ V_i = ∅. Then D is a transversal of G of the χ – Partition Π = { V_1 ∪{v}, V_2,...., V_χ} of G. Note that v ∉ D. Also D do not contain any neighbour of v as otherwise Υ_{tstd} (G) ≤ Υ_{tstd} (G \ {v}). Trivially S = D ∪ {u} is the Total Dominating Color Transversal Set of G for some u ∈ N (v). So Υ_{tstd} (G) ≤ Υ_{tstd} (G \ {v}) + 1. Therefore by Υ_{tstd} (G \ {v}) < Υ_{tstd} (G) we have Υ_{tstd} (G \ {v}) + 1 = Υ_{tstd} (G). Hence S is a Υ_{tstd} - Set of G not containing v. Also note that u ∈ N (v) such that u ∈ χ^T(w, S, Π), for some vertex w (≠ u) and χ – Partition Π of G for which S is a transversal. Hence (1) and (2).

Conversely assume (1) and (2).

Let D = S \ {u}.

Claim: D is a Total Dominating Color Transversal of G \ {v}.

Note that D is a transversal of G \ {v}. Also D has no isolates.

Let x ∈ V \ {v}.

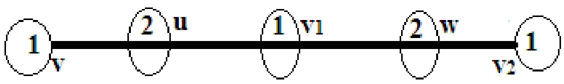
If x = u then as S is a Total Dominating Set of G there exists y ∈ S \ {u} = D, such that y is adjacent to x. So let as assume x ≠ u. If there does not exists any y ∈ S \ {u} = D such that x is adjacent to y then x is adjacent to u only in S. Hence D = S \ {u} has isolates, which is contradiction. Hence there exists some y ∈ S \ {u} = D such that it is adjacent to x. Therefore D is Total Dominating Color Transversal Set of G \ {v}.

Therefore we have Υ_{tstd} (G \ {v}) ≤ |D| < |S| = Υ_{tstd} (G). Hence v ∈ V^-_{tstd} .

Corollary 3.8: Let G = (V, E) be a graph and v ∈ V with v ∉ V^i , {v} is not a color class of any χ - Partition of G and deg (v) < χ. If v ∈ V^-_{tstd} then Υ_{tstd} (G \ {v}) + 1 = Υ_{tstd} (G).

Proof: It is obvious by the construction of set S, in the 'if' part, in above theorem 3.7.

Example 3.9:



G = P_5

Fig. 4

S = Υ_{tstd} - Set of G = {u, v_1, w} and u ∈ χ^T(w, S, Π). D = Υ_{tstd} - Set of G \ {v} = {v_1, w}. Hence Υ_{tstd} (G) = 3 and Υ_{tstd} (G \ {v}) = 2. Note that deg (v) = 1 < χ (G)= 2.

Theorem 3.10: Let G = (V, E) be a graph and v ∈ V with v ∉ V^i and {v} is not a color class of any χ - Partition of G. If v ∈ V^+_{tstd} then the following two conditions are satisfied: (1) Every Υ_{tstd} – Set of G contains v.

(2) If $S \subset V \setminus N[v]$ such that $|S| = \Upsilon_{tstd}$ (G), then S is not a Total Dominating Color Transversal Set of $G \setminus \{v\}$.

Proof: As{v} is not a color class of any χ - Partition of G, $\chi (G \setminus \{v\}) = \chi (G)$. In such case a Total Dominating Color Transversal Set, not containing v, of G is also transversal of some χ - Partition of $G \setminus \{v\}$.

Assume $v \in V^{+}_{tstd}$.

Suppose (1) is not true. Then there exists a Υ_{tstd} – Set, say D, of G not containing v. Then D is also a Total Dominating Color Transversal Set of $G \setminus \{v\}$. Hence we have $\Upsilon_{tstd} (G \setminus \{v\}) \leq \Upsilon_{tstd}$ (G), which is contradiction. So (1) is true.

Suppose (2) is not true. If $S \subset V \setminus N[v]$ such that $|S| = \Upsilon_{tstd}$ (G), then S is a Total Dominating Color Transversal Set of $G \setminus \{v\}$. So $\Upsilon_{tstd} (G \setminus \{v\}) \leq \Upsilon_{tstd} (G) = |S|$, which is contradiction. So (2) is true.

Theorem 3.11: Let G = (V, E) be a graph and $v \in V$ with $v \notin V^{i}$ and {v} is not a color class of any χ - Partition of G and deg (v) < χ. Then $v \in V^{+}_{tstd}$ if the following two conditions are satisfied:
(1) Every Υ_{tstd} – Set of G contains v.

(2) If $S \subset V \setminus N[v]$ such that $|S| = \Upsilon_{tstd}$ (G), then S is not a Total Dominating Color Transversal Set of $G \setminus \{v\}$.

Proof: Assume (1) and (2).

It is enough to show that $v \notin V^{-}_{tstd}$ and $v \notin V^{0}_{tstd}$.

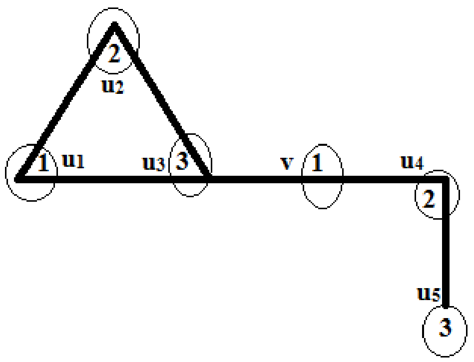
If $v \in V^{-}_{tstd}$ then by theorem 3.7, there exists a Υ_{tstd} - Set of G not containing v, which is a contradiction. So let us assume that $v \in V^{0}_{tstd}$. Then $\Upsilon_{tstd} (G \setminus \{v\}) = \Upsilon_{tstd}$ (G). Let D be a Υ_{tstd} – Set of $G \setminus \{v\}$. Then $v \notin D$. Note that $|D| = \Upsilon_{tstd}$ (G). So if $D \subset V \setminus N[v]$ then by (2), D cannot be a Total Dominating Color Transversal of $G \setminus \{v\}$. So there exists $u \in N(v)$ such that $u \in D$. So D is a Total Dominating Set of G. Also deg (v) < χ , then as in theorem 3.7, D is a transversal of some χ – Partition of G. Therefore D is a Total Dominating Color Transversal Set of G. Hence D is a Υ_{tstd} - Set of G as $|D| = \Upsilon_{tstd}$ (G), which is contradiction to (1) as $v \notin D$. Therefore $v \notin V^{0}_{tstd}$.

Theorem 3.12: Let G = (V, E) be a graph and $v \in V$ with $v \notin V^{i}$ and {v} is not a color class of any χ - Partition of G and deg (v) < χ . Then $v \in V^{+}_{tstd}$ if and only if the following two conditions hold:
(1) Every Υ_{tstd} – Set of G contains v.

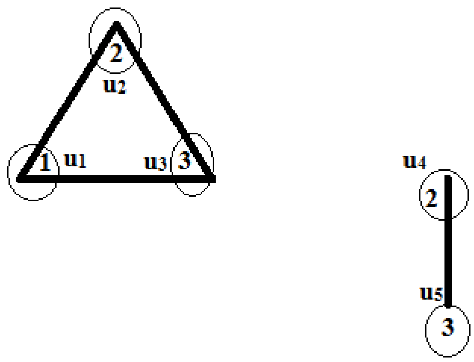
(2) If $S \subset V \setminus N[v]$ such that $|S| = \Upsilon_{tstd}$ (G), then S is not a Total Dominating Color Transversal Set of $G \setminus \{v\}$.

Proof: Obvious by theorem 3.10 and theorem 3.11.

Example 3.13:



G
Fig. 5



$G \setminus \{v\}$
Fig. 6

Υ_{tstd} (G) = 3 and $\Upsilon_{tstd} (G \setminus \{v\}) = 4$.

4. CONCLUSION

Throughout this paper we removed only those vertices that do not form color class of any χ - Partition of a graph G. Removal of any vertex, in general, can also be discussed in future. Definitely this will not be easy. Much more amount of work is required for this because of the fact that Total Dominating Color Transversal number is an amalgam of Proper coloring and Total Domination.

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