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S-METRIC SPACES, EXPANDING MAPPINGS & FIXED POINT THEOREMS

Suraj Shrivastava[1], R. D. Daheriya[2], Manoj Ughade[3]

[1]Department of Mathematics, Swami Vivekanand College of Science and Technology, Bhopal, Madhya Pradesh, India

[2]Department of Mathematics, Government J.H. Post Graduate College, Betul, Madhya Pradesh, India

[3]Department of Mathematics, Sarvepalli Radhakrishnan University, Bhopal, Madhya Pradesh, India

Correspondence should be addressed to Suraj Shrivastava: *sshelpyou@gmail.com*

Abstract: *Sedghi et al. [28] introduced S-metric space and established some fixed point theorems for a self-mapping on a complete S-metric space. In the present paper, we prove some fixed point theorems for surjection satisfying various expansive type conditions in the setting of a S-metric space. The presented theorems extend, generalize and improve many existing results in the literature.*

Keywords: *S-metric spaces, surjection, expansive mapping, fixed point.*

1. INTRODUCTION

Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. For example, Gahler [11] and Dhage [7] introduced the concepts of 2-metric spaces and *D*-metric spaces, respectively. Mustafa and Sims [22] introduced a new structure of generalized metric spaces which are called *G*-metric spaces as a generalization of metric spaces (X, d) to develop and introduce a new fixed point theory for various mappings in this new structure. Sedghi et al. [27] have introduced D^*-metric spaces which is a probable modification of the definition of *D*-metric spaces introduced by Dhage [7] and proved some basic properties in D^*-metric spaces, (see [27, 29]).

Recently, Sedghi et al. [28] have introduced S-metric space. The S-metric space is a space with three dimensions. The study of expansive mappings is very interesting research area of fixed point theory. The study of expansive mappings is a very interesting research area in fixed point theory. In 1984, Wang et.al [32] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko [3] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. For more details on expanding mapping and related results we refer the reader to [1, 4, 14-15, 31, 33-34].

In our paper, we work in S-metric space. Also, most of these results, under different expansive type conditions, use surjective mappings. These results improve and generalized some important known results.

2. PRELIMINARIES

Following definitions and fundamental results are required for our further use.

In 1963, Gähler [11] introduced the notion of a 2-metric space as follows.

Definition 2.1 Let X be a nonempty set. A function $d: X^3 \to [0, +\infty)$ is said to be a 2-metric on X if the following conditions hold:

(d1). For any distinct points $x, y \in X$ there is $z \in X$ such that $d(x, y, z) \neq 0$;

(d2). $d(x, y, z) = 0$ if any two elements of the set $\{x, y, z\}$ in X are equal;

(d3). $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(z,x,y) = d(y,z,x) = d(z,y,x)$,

(d4). $d(x,y,z) \leq d(x,y,a) + d(x,a,z) + d(a,y,z)$ for all $x,y,z,a \in X$.

The pair (X,d) is called a 2-metric space.

Gähler [11] claimed that 2-metric space is a generalization of an ordinary metric space. He mentioned in [12] that $d(x,y,z)$ geometrically represents the area of a triangle formed by the points $x, y, z \in X$ as its vertices. On the other hand, Ha et al. [13] and Sharma [30] found some mathematical flaws in these claims. It was demonstrated in [30] that $d(x,y,z)$ does not always represent the area of a triangle formed by the points $x,y,z \in X$. Ha et al. [13] proved that the 2-metric is not sequentially continuous in each of its arguments whereas an ordinary metric satisfies this property.

In order to carry out meaningful studies of fixed point results, Dhage [7] suggested an improvement in the basic structure of 2-metric space.

In 1984, Dhage in his Ph.D. thesis [5] identified condition (d2) as a weakness in Gähler's theory of a 2-metric space. To overcome these problems, he then introduced the concept of a *D*-metric space.

Definition 2.2 Let X be a nonempty set. A function $D: X^3 \to [0,+\infty)$ is said to be a D-metric on X if the following conditions hold:

(D1). $D(x,y,z) \geq 0$ for all $x,y,z \in X$ and equality holds if and only if $x = y = z$;

(D2). $D(x,y,z) = D(x,z,y) = D(y,x,z) = D(z,x,y) = D(y,z,x) = D(z,y,x)$,

(D3). $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x,y,z,a \in X$.

The pair (X,D) is called a D-metric space.

It is important to note that condition (d3) and (D2) are equivalent. Condition (d4) and (D3) are also equivalent, whereas (d1) and (d2) have been replaced by (D1). Dhage [7] modified condition (d2) to obtain the natural non-negativity condition of ordinary metric. Dhage [6] then studied topological properties of *D*-metric space in a series of papers. Naidu et al. [25] proved that the concepts of convergent sequences and sequential continuity are not well defined in *D*-metric spaces. Naidu et al. [24] pointed out some drawbacks in the idea of open balls in *D*-metric space. In 2003, Mustafa and Sims [23] identified condition (D3) as a weakness in Dhage's theory of *D*-metric space.

In 2006, Mustafa and Sims [22] introduced the notion of *G*-metric space and suggested an important generalization of metric space as follows.

Definition 2.3 Let X be a non-empty set. Suppose that a mapping $G: X^3 \to [0,+\infty)$ satisfies:

(G1). $G(x,y,z) = 0$ if $x = y = z$;

(G2). $0 < G(x,x,y), \forall\, x,y \in X$ with $x \neq y$;

(G3). $G(x,x,y) \leq G(x,y,z), \forall\, x,y,z \in X$ with $y \neq z$;

(G4). $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(z,x,y) = G(y,z,x) = G(z,y,x)$;

(G5). $G(x,y,z) \leq G(x,a,a) + G(a,y,z), \forall\, x,y,z \in X$ (Rectangle inequality).

Then the pair (X,G) is called a generalized G-metric space or, more specifically, a Gmetric space.

Note that condition (D1) has been replaced with (G1), (G2), and (G3). Condition (D2) is equivalent to (G4) and condition (D3) has been replaced by (G5). The deficiency of Dhage's theory of *D*-metric is thus corrected. Subsequently, Mustafa and Sims [22] studied some topological properties of *G*-metric space and afterwards some authors have obtained generalized fixed point theorems in the setup of *G*-metric space; see for example [21].

Unlike in the theory of *G*-metric space, where condition (D1) was replaced with the three separate axioms (G1), (G2), and (G3), Sedghi et al. [27] observed that condition (D1) can be replaced with two axioms and thus introduced the notion of a D^*-metric space as follows.

Definition 2.4 Let X be a non-empty set. An D^*-metric on X is a function $D^*: X^3 \to [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(D*1). $D^*(x, y, z) \geq 0$;

(D*2). $D^*(x, y, z) = 0$ if and only if $x = y = z$;

(D*3). $D^*(x, y, z) = D^*(x, z, y) = D^*(y, x, z) = D^*(z, x, y) = D^*(y, z, x) = D^*(z, y, x)$

(D*4). $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

Then D^* is called an D^*-metric on X and (X, D^*) is called an D^*-metric space.

Note that condition (D1) has been replaced with (D^*1) and (D^*2). Condition (D2) and (D^*3) are equivalent. Condition (D3) has been replaced with (D^*4). The tetrahedral inequality in *D*-metric has been replaced with the prototypical rectangular inequality adopted by Mustafa and Sims [22]. Every *G*-metric space is a D^*-metric space. Indeed conditions (G1), (G2), and (G3) imply (D^*1). Axioms (G1) and (D^*2) are equivalent. (G4) and (D^*4) are also equivalent, whereas (G4) and (G5) imply (D^*4). The converse, however, is false in general; a D^*-metric space is not necessarily a *G*-metric space.

Sedghi et al. [28] identified condition (G3) as a peculiar limitation of the *G*-metric space but classified the symmetry condition as a common weakness of both *G*- and D^*-metric spaces. To overcome these difficulties, Sedghi et al. [28] introduced a new generalized metric space called an *S*-metric space.

Definition 2.5 Let X be a non-empty set. An S-metric on X is a function $S: X^3 \to [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(S1). $S(x, y, z) \geq 0$;

(S2). $S(x, y, z) = 0$ if and only if $x = y = z$;

(S3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and (X, S) is called an S-metric space.

The following is the intuitive geometric example for S-metric spaces.

Example 2.6 ([28], Example 2.4) Let $X = \mathbb{R}^2$ and d be the ordinary metric on X. Put $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

Lemma 2.7 ([28], Lemma 2.5) Let (X, S) be an S-metric space. Then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

Lemma 2.8 ([9], Lemma 1.6) Let (X, S) be an S-metric space. Then $S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$ and $S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$ for all $x, y, z \in X$.

Definition 2.9 ([28]) Let (X, S) be an S-metric space.

1. A sequence $\{x_n\}_{n=1}^{\infty}$ is called convergent to x in (X, S), written $\lim_{n\to+\infty} x_n = x$, if $\lim_{n\to+\infty} S(x_n, x_n, x) = 0$.
2. A sequence $\{x_n\}_{n=1}^{\infty}$ is called Cauchy in (X, S) if $\lim_{n\to+\infty} S(x_n, x_n, x_m) = 0$.
3. (X, S) is called complete if every Cauchy sequence in (X, S) is a convergent sequence in (X, S).

From [28, Examples in page 260], we have the following.

Example 2.10

1. Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$, is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete.

2. Let Y be a non-empty set of $\mathbb{R}$. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in Y$, is an S-metric on Y. If Y is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space Y is complete.

Lemma 2.11 ([28], Lemma 2.11) Let (X,S) be an S-metric space. If the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to x, then x is unique.

Lemma 2.12 ([28], Lemma 2.12) Let (X,S) be an S-metric space. If $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, then $\lim_{n\to+\infty} S(x_n, x_n, y_n) = S(x,x,y)$.

Remark 2.13 [9] It is easy to see that every D^*-metric is S-metric, but in general the converse is not true, see the following example.

Example 2.14 [9] Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is S-metric on X, but it is not D^*-metric because it is not symmetric.

The following lemma shows that every metric space is an S-metric space.

Lemma 2.15 ([9], Lemma 1.10) Let (X,d) be a metric space. Then we have

1. $S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$ is an S-metric on X.
2. $\lim_{n\to+\infty} x_n = x$ in (X,d) if and only if $\lim_{n\to+\infty} x_n = x$ in (X,S_d).
3. $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X,d) if and only if $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X,S_d).
4. (X,d) is complete if and only if (X,S_d) is complete.

The following example ([10], example 1.9) proves that the inversion of Lemma 2.15 does not hold.

Example 2.16 ([9], Example 1.10) Let $X = \mathbb{R}$ and let $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in X$. By [28, Example (1), page 260], (X,S) is an S-metric space. Dung et al. [10] proved that there does not exist any metric d such that $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$. Indeed, suppose to the contrary that there exists a metric d with $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$.Then $d(x,z) = \frac{1}{2}S(x,x,z) = 2|x-z|$ and $d(x,y) = \frac{1}{2}S(x,y,y) = 2|x-y|$ for all $x,y,z \in X$. It is a contradiction.

In 2012, Sedghi et al. [28] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [28, Remarks 1.3] and [28, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [10] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct.

Example 2.17 (see [10]) There exists a G-metric which is not an S-metric.

Proof Let X be the G-metric space in ([22], Example 1). Then we have

$$2 = G(a,b,b) > 1 = G(a,a,b) + G(b,b,b) + G(b,b,b).$$

This proves that G is not an S-metric on X.

Example 2.18 (see [10]) There exists an S-metric which is not a G-metric.

Proof Let (X,S) be the S-metric space in Example 2.16. We have

$$S(1,0,2) = |0+2-2| + |0-2| = 2.$$

$$S(2,0,1) = |0+1-4| + |0-1| = 4.$$

Then $S(1,0,2) \neq S(2,0,1)$ This proves that S is not a G-metric.

3. Main Result

We begin with following some lemmas.

Lemma 3.1 Let (X,S) be a S-metric space and let $\{x_k\}_{k=0}^{n} \subset X$. Then

$$S(x_0,x_0,x_n) \leq 2\sum_{i=0}^{n-2} S(x_i,x_i,x_{i+1}) + S(x_{n-1},x_{n-1},x_n) \tag{3.1}$$

Proof: By the third condition of S-metric, we get

$$S(x_0, x_0, x_n) \leq 2S(x_0, x_0, x_1) + S(x_n, x_n, x_1) \tag{3.2}$$

Also,

$$\begin{aligned} S(x_n, x_n, x_1) &\leq S(x_n, x_n, x_n) + S(x_n, x_n, x_n) + S(x_1, x_1, x_n) \\ &= S(x_1, x_1, x_n) \end{aligned}$$

and

$$\begin{aligned} S(x_1, x_1, x_n) &\leq S(x_1, x_1, x_1) + S(x_1, x_1, x_1) + S(x_n, x_n, x_1) \\ &= S(x_n, x_n, x_1) \end{aligned}$$

Combining above two inequalities, we obtain

$$S(x_1, x_1, x_n) = S(x_n, x_n, x_1)$$

Thus, from (3.2), we have

$$S(x_0, x_0, x_n) \leq 2S(x_0, x_0, x_1) + S(x_1, x_1, x_n)$$

Again

$$\begin{aligned} S(x_1, x_1, x_n) &\leq 2S(x_1, x_1, x_2) + S(x_n, x_n, x_2) \\ &\leq 2S(x_1, x_1, x_2) + S(x_2, x_2, x_n) \end{aligned}$$

Hence

$$S(x_0, x_0, x_n) \leq 2S(x_0, x_0, x_1) + 2S(x_1, x_1, x_2) + S(x_2, x_2, x_n)$$

Proceeding this way, we have

$$S(x_0, x_0, x_n) \leq 2\sum_{i=0}^{n-2} S(x_i, x_i, x_{i+1}) + S(x_{n-1}, x_{n-1}, x_n)$$

From Lemma 3.1, we deduce the following result.

Lemma 3.2 Let (X, S) be a S-metric space and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X such that

$$S(x_n, x_n, x_{n+1}) \leq \lambda S(x_{n-1}, x_{n-1}, x_n) \tag{3.3}$$

where $\lambda \in [0, 1)$ and $n = 1, 2, \ldots$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, S).

Proof From (3.3), by induction, we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq \lambda S(x_{n-1}, x_{n-1}, x_n) \\ &\leq \lambda^2 S(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\leq \cdots \leq \lambda^n S(x_0, x_0, x_1) \end{aligned} \tag{3.4}$$

Let $m > n$. It follows that

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2\sum_{i=n}^{m-2} \lambda^i S(x_0, x_0, x_1) + \lambda^{m-1} S(x_0, x_0, x_1) \\ &\leq 2\lambda^n [1 + \lambda + \lambda^2 + \cdots \ldots \ldots] S(x_0, x_0, x_1) \\ &\leq \frac{2\lambda^n}{1-\lambda} S(x_0, x_0, x_1) \end{aligned} \tag{3.5}$$

It is noted that $\lambda < 1$. Assume that $S(x_0, x_0, x_1) > 0$. By taking limit as $m, n \to +\infty$ in above inequality we get

$$lim_{n,m \to +\infty} S(x_n, x_m, x_m) = 0. \tag{3.6}$$

So $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Also, if $S(x_0, x_0, x_1) = 0$, then $S(x_n, x_m, x_m) = 0$ for all $m > n$ and hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.

Now, our first main results as follows.

Theorem 3.3 Let (X, S) be a complete S-metric space and $T: X \to X$ be a surjection. Suppose that there exist a constant $a > 1$ such that

$$S(Tx, Tx, Ty) \geq a\, S(x, x, y) \tag{3.7}$$

$\forall\, x, y \in X$. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Since T is onto, there is an element $x_1 \in X$ satisfying $x_1 \in T^{-1}(x_0)$. By the same way, we can find $x_n \in T^{-1}(x_{n-1})$ for $n = 2,3,4, \ldots..$ If $S(x_{m-1}, x_{m-1}, x_m) = 0$ for some m, then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T. Without loss of generality, we can suppose that $S(x_{n-1}, x_{n-1}, x_n) > 0$, that is, $x_n \neq x_{n-1}$ for every n. From (3.7), we have

$$\begin{aligned} S(x_{n-1}, x_{n-1}, x_n) &= S(Tx_n, Tx_n, Tx_{n+1}) \\ &\geq a\, S(x_n, x_n, x_{n+1}) \end{aligned}$$

So, it must be the case that

$$S(x_n, x_n, x_{n+1}) \leq \frac{1}{a} S(x_{n-1}, x_{n-1}, x_n) \tag{3.9}$$

where $\frac{1}{a} < 1$.

Let $\lambda = \frac{1}{a}$. Then $0 < \lambda < 1$ and

$$S(x_n, x_n, x_{n+1}) \leq \lambda S(x_{n-1}, x_{n-1}, x_n) \tag{3.10}$$

By Lemma 3.2, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (X, S), there exists $x^* \in X$ such that $x_n \to x^*$. Now T is surjection. So there exists a point $p \in X$ such that $p \in T^{-1}(x^*)$ and so $x^* = Tp$. Consider from (3.7), we have

$$\begin{aligned} S(x_n, x_n, x^*) &= S(Tx_{n+1}, Tx_{n+1}, Tp) \\ &\geq aS(x_{n+1}, x_{n+1}, p) \end{aligned} \tag{3.11}$$

Taking the limit as $n \to +\infty$, we have

$$0 \geq aS(x^*, x^*, p)$$

which implies that $S(x^*, x^*, p) = 0$, since $a > 1$. Therefore $p = x^*$ and hence $Tx^* = x^*$.

Now, we show that uniqueness. Suppose that $x^\star \neq y^\star$ is also another fixed point of T, then from condition (3.7), we obtain

$$\begin{aligned} S(x^\star, x^\star, y^\star) &= S(Tx^\star, Tx^\star, Ty^\star) \\ &\geq \mathrm{a}S(x^\star, x^\star, y^\star) \end{aligned}$$

and therefore $S(x^\star, x^\star, y^\star) = 0$. So $x^\star = y^\star$. This completes the proof.

Corollary 3.4 Let (X, S) be a complete S-metric space and $T: X \to X$ be a surjection. Suppose that there exist a positive integer n and a real number $a > 1$ such that

$$S(T^n x, T^n x, T^n y) \geq a\, S(x, x, y) \tag{3.12}$$

$\forall\, x, y \in X$. Then T has a unique fixed point in X.

Proof From Theorem 3.3, T^n has a fixed point $x^\star$. But $T^n(Tx^\star) = T(T^n x^\star) = Tx^\star$, So $Tx^\star$ is also a fixed point of T^n. Hence $Tx^\star = x^\star, x^\star$ is a fixed point of T. Since the fixed point of T is also fixed point of T^n , the fixed point of T is unique.

Theorem 3.5 Let (X, S) be a complete S-metric space. Assume that the mapping $T : X \to X$ is surjective and satisfies the condition

$$S(Tx, Tx, Ty) \geq aS(x, x, y) + bS(x, x, Tx) + cS(y, y, Ty) \tag{3.13}$$

where a, b, c, d are non-negative constants with $a + b + c > 1$. Then T has a fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary. Similar to the proof of Theorem 3.2, we can obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in T^{-1}(x_{n-1})$ for $n = 2,3, \ldots$ If $S(x_{m-1}, x_{m-1}, x_m) = 0$ for some m, then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T. Without loss of generality, we can suppose that $S(x_{n-1}, x_{n-1}, x_n) > 0$, that is, $x_n \neq x_{n-1}$ for every n. From (3.13), we have

$$\begin{aligned} S(x_{n-1}, x_{n-1}, x_n) &= S(Tx_n, Tx_n, Tx_{n+1}) \\ &\geq aS(x_n, x_n, x_{n+1}) + bS(x_n, x_n, Tx_n) + cS(x_{n+1}, x_{n+1}, Tx_{n+1}) \\ &= aS(x_n, x_n, x_{n+1}) + bS(x_n, x_n, x_{n-1}) + cS(x_{n+1}, x_{n+1}, x_n) \end{aligned}$$

Since $S(x_n, x_n, x_{n-1}) = S(x_{n-1}, x_{n-1}, x_n)$ and $S(x_{n+1}, x_{n+1}, x_n) = S(x_n, x_n, x_{n+1})$, therefore

$$S(x_{n-1}, x_{n-1}, x_n) \geq aS(x_n, x_n, x_{n+1}) + bS(x_{n-1}, x_{n-1}, x_n) + cS(x_n, x_n, x_{n+1})$$

So, it must be the case that

$$(1-b)S(x_{n-1}, x_{n-1}, x_n) \geq (a+c)S(x_n, x_n, x_{n+1}) \tag{3.14}$$

If $a + c = 0$, then $b > 1$. The above inequality implies that a negative number is greater than or equal to zero. That is impossible. So, $a + c \neq 0$ and $1 - b > 0$. Therefore,

$$S(x_n, x_n, x_{n+1}) \leq \frac{1-b}{a+c} S(x_{n-1}, x_{n-1}, x_n) \tag{3.15}$$

where $0 < \frac{1-b}{a+c} < 1$. Let $\lambda = \frac{1-b}{a+c}$. Then $0 < \lambda < 1$ and

$$S(x_n, x_n, x_{n+1}) \leq \lambda S(x_{n-1}, x_{n-1}, x_n) \tag{3.16}$$

By Lemma 3.2, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By completeness of (X, S), there exists $x^* \in X$ such that $x_n \to x^*$. Now T is surjective mapping. So there exists a point $p \in X$ such that $p \in T^{-1}(x^*)$ and so $x^* = Tp$. Consider from (3.13), we have

$$\begin{aligned} S(x_n, x_n, x^*) &= S(Tx_{n+1}, Tx_{n+1}, Tp) \\ &\geq aS(x_{n+1}, x_{n+1}, p) + bS(x_{n+1}, x_{n+1}, Tx_{n+1}) + cS(p, p, Tp) \\ &\geq aS(x_{n+1}, x_{n+1}, p) + bS(x_{n+1}, x_{n+1}, x_n) + cS(p, p, x^*) \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$\begin{aligned} 0 &\geq aS(x^*, x^*, p) + bS(x^*, x^*, x^*) + cS(p, p, x^*) \\ &= (a+c)S(p, p, x^*) = (a+c)S(x^*, x^*, p) \end{aligned}$$

So,

$$0 \geq (a+c)S(p, p, x^*).$$

which implies that $S(p, p, x^*) = 0$, since $a + c \neq 0$. Therefore $p = x^*$ and hence $Tx^* = x^*$.

Remark 3.6 Setting $b = c = 0$ in Theorem 3.4, we can obtain the Theorem 3.2.

Theorem 3.7 Let (X, S) be a complete S-metric space and $T: X \to X$ is a continuous surjection. Suppose that there exists a constant $k > 1$ such that

$$S(Tx, Tx, Ty) \geq ku, \text{ for some } u \in \{S(x, x, y), S(x, x, Tx), S(y, y, Ty)\} \tag{3.17}$$

$\forall\, x, y \in X$. Then T has a fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Similar to the proof of Theorem 3.2, we can obtain a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in T^{-1}(x_{n-1})$ for $n = 2,3,\ldots$ If $S(x_{m-1}, x_{m-1}, x_m) = 0$ for some m, then $x_{m-1} = x_m$ and $x_m \in T^{-1}(x_{m-1})$ implies $Tx_m = x_{m-1} = x_m$ and so x_m is a fixed point of T. Without loss of generality, we can suppose that $S(x_{n-1}, x_{n-1}, x_n) > 0$, that is, $x_n \neq x_{n-1}$ for every n. From (3.17), we have

$$S(x_{n-1}, x_{n-1}, x_n) = S(Tx_n, Tx_n, Tx_{n+1},) \geq \lambda u_n \tag{3.18}$$

where
$$\begin{aligned} u_n &\in \{S(x_n, x_n, x_{n+1}), S(x_n, x_n, Tx_n), S(x_{n+1}, x_{n+1}, Tx_{n+1})\} \\ &= \{S(x_n, x_n, x_{n+1}), S(x_n, x_n, x_{n-1}), S(x_{n+1}, x_{n+1}, x_n)\} \\ &= \{S(x_n, x_n, x_{n+1}), S(x_{n-1}, x_{n-1}, x_n), S(x_n, x_n, x_{n+1})\} \end{aligned}$$

$$= \{S(x_n, x_n, x_{n+1}), S(x_{n-1}, x_{n-1}, x_n)\}$$

Now we have to consider the following three cases.

If $u_n = S(x_{n-1}, x_{n-1}, x_n)$, then

$$S(x_{n-1}, x_{n-1}, x_n) \geq \lambda S(x_n, x_{n-1}, x_{n-1})$$

which implies that $S(x_{n-1}, x_{n-1}, x_n) = 0$, that is, $x_{n-1} = x_n$. This is a contradiction.

If $u_n = S(x_n, x_n, x_{n+1})$, then

$$S(x_{n-1}, x_{n-1}, x_n) \geq kS(x_n, x_n, x_{n+1})$$

and so

$$S(x_n, x_n, x_{n+1}) \leq \frac{1}{k} S(x_{n-1}, x_{n-1}, x_n)$$

Let $\lambda = \frac{1}{k} < 1$.Then

$$S(x_n, x_n, x_{n+1}) \leq \lambda S(x_{n-1}, x_{n-1}, x_n)$$

by Lemma 3.2, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, S) is a complete, the sequence $\{x_n\}_{n=1}^{\infty}$ is converges to a point $x^\star \in X$. Since T is continuous, it is clear that $x^\star$ is a fixed point of T. This completes the proof.

Example 3.8 Let $X = \mathbb{R}$, then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. By [28, Example (1), page 260], (X, S) is an S-metric space. Define $T: X \to X$ by $Tx = \frac{5}{2}x$. Obviously, T is continuous surjective map on X. Now

$$\begin{aligned} S(Tx, Tx, Ty) &= 2|Tx - Ty| \\ &= 5[|x - y|] \\ &\geq aS(x, x, y) \end{aligned}$$

So for a $= \frac{12}{5} > 1$ and all the conditions of Theorem 3.2 are satisfied. Therefore $x^\star = 0$ is the unique fixed point of T.

4. Common Fixed Point Theorems

Now, we give a common fixed point theorem of two weakly compatible mappings in S-metric spaces.

In [17] Jungck introduced the concept of commuting maps. In [18] Jungck introduced the concept of compatible mappings which generalize the concept of commuting maps. Jungck in [19] further generalized the concept of weakly compatible maps as follows.

Let S and T be two self-mappings on a nonempty set X. Then S and T are said to be weakly compatible if they commute at all of their coincidence points; that is, $Sx = Tx$ for some $x \in X$ and then $STx = TSx$.

Theorem 4.1 Let (X, S) be a complete S-metric space. Let f and T be two self-mappings of X and $T(X) \subseteq f(X)$. Suppose that there exists a constant $k > 1$ such that

$$S(fx, fx, fy) \geq kS(Tx, Tx, Ty) \tag{4.1}$$

$\forall\, x, y \in X$. If one of the subspaces $T(X)$ or $f(X)$ is complete, then f and T have a unique point of coincidence in X. Moreover, if f and T are weakly compatible, then f and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$. Since $T(X) \subseteq f(X)$, choose $x_1 \in X$ such that $y_1 = fx_1 = Tx_0$. In general, choose $x_{n+1} \in X$ such that $y_{n+1} = fx_{n+1} = Tx_n$. Now by (4.1), we have

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &= S(fx_n, fx_n, fx_{n+1}) \\ &\geq kS(Tx_n, Tx_n, Tx_{n+1}) = kS(y_{n+1}, y_{n+1}, y_{n+2}) \end{aligned}$$

and so

$$S(y_{n+1}, y_{n+1}, y_{n+2}) \leq \frac{1}{k} S(y_n, y_n, y_{n+1}) = \lambda S(y_n, y_n, y_{n+1}) \tag{4.2}$$

where $= \frac{1}{k} < 1$. Then by Lemma 3.2, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $T(X) \subseteq f(X)$ and $T(X)$ or $f(X)$ is a complete subspace of X. Then $(f(X), S)$ is complete S-metric space and so the sequence $\{y_n\} = \{Tx_{n-1}\} \subseteq f(X)$ is converges in the S-metric space $(f(X), S)$, that is, there exists $z^\star \in X$ such that

$$\lim_{n\to+\infty} S(y_n, y_n, z^\star) = 0.$$

Consequently, we can find $u \in X$ such that $fu = z^\star$. Now to show that $Tu = z^\star$. From (4.1), we have

$$S(Tx_n, Tx_n, Tu) \leq \frac{1}{k} S(fx_n, fx_n, fu) \tag{4.3}$$

Taking limit as $n \to +\infty$ in the above inequality, we get

$$S(z^\star, z^\star, Tu) \leq \frac{1}{k} S(z^\star, z^\star, fu) = 0,$$

This implies that $S(z^\star, z^\star, Tu) = 0$ and so $Tu = z^\star$. Therefore, $fu = Tu = z^\star$. Since f and T be weakly compatible, $fTu = Tfu$, that is, $fz^\star = Tz^\star$.

Now we show that $z^\star$ is a common fixed point of f and T. From condition (4.1)

$$S(fx_n, fx_n, fz^\star) \geq kS(Tx_n, Tx_n, Tz^\star,)$$

Proceeding to the limit as $n \to +\infty$, we have $S(z^\star, z^\star, fz^\star) \geq kS(z^\star, z^\star, Tz^\star) = kS(z^\star, z^\star, fz^\star)$, which implies that $S(z^\star, z^\star, fz^\star) = 0$. Hence $fz^\star = z^\star$ and so $fz^\star = Tz^\star = z^\star$.

Finally, assume $z^\star \neq w^\star$ is also another common fixed point of f and T. From (4.1), we get

$$\begin{aligned} S(z^\star, z^\star, w^\star) &= S(fz^\star, fz^\star, fw^\star) \\ &\geq kS(Tz^\star, Tz^\star, Tw^\star) \\ &= kS(z^\star, z^\star, w^\star) \end{aligned}$$

This is true only when $S(z^\star, z^\star, w^\star) = 0$. So $z^\star = w^\star$. Hence f and T have a unique fixed point in X. This completes the proof.

Theorem 4.2 Let (X, S) be a complete S-metric space. Let f and T be two self-mappings of X and $T(X) \subseteq f(X)$. Suppose that $a, b, c \geq 0$ with $a + b + c > 1$ such that

$$S(fx, fx, fy) \geq aS(Tx, Tx, Ty) + bS(Tx, Tx, fx) + cS(Ty, Ty, fy) \tag{4.4}$$

$\forall\, x, y \in X$. If one of the subspaces $T(X)$ or $f(X)$ is p_b-complete, then f and T have a point of coincidence in X.

Moreover, if $a > 1$, then point of coincidence is unique. If f and T be weakly compatible and $a > 1$, then f and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$. Since $T(X) \subseteq f(X)$, choose $x_1 \in X$ such that $y_1 = fx_1 = Tx_0$. In general, choose $x_{n+1} \in X$ such that $y_{n+1} = fx_{n+1} = Tx_n$. Now by (4.4), we have

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &= S(fx_n, fx_n, fx_{n+1}) \\ &\geq aS(Tx_n, Tx_n, Tx_{n+1}) + bS(Tx_n, Tx_n, fx_n) \\ &\quad + cS(Tx_{n+1}, Tx_{n+1}, fx_{n+1}) \\ &= aS(y_{n+1}, y_{n+1}, y_{n+2}) + bS(y_{n+1}, y_{n+1}, y_n) \\ &\quad + cS(y_{n+2}, y_{n+2}, y_{n+1}) \\ &= aS(y_{n+1}, y_{n+1}, y_{n+2}) + bS(y_n, y_n, y_{n+1}) \\ &\quad + cS(y_{n+1}, y_{n+1}, y_{n+2}) \text{ (as } S(x, x, y) = S(y, y, x)) \end{aligned}$$

and so

$$(1 - b)S(y_n, y_n, y_{n+1}) \geq (a + c)S(y_{n+1}, y_{n+1}, y_{n+2})$$

If $a+c=0$, then $b>1$. The above inequality implies that a negative number is greater than or equal to zero. That is impossible. So, $a+c\neq 0$ and $1-b>0$. Therefore,

$$S(y_{n+1},y_{n+1},y_{n+2})\leq \lambda S(y_n,y_n,y_{n+1}) \tag{4.5}$$

where $\lambda=\frac{1-b}{a+c}<1$. Then by Lemma 3.2, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $T(X)\subseteq f(X)$ and $T(X)$ or $f(X)$ is a complete subspace of X. Then, $(f(X),S)$ is complete S-metric space and so the sequence $\{y_n\}=\{Tx_{n-1}\}\subseteq f(X)$ is converges in the S-metric space $(f(X),S)$, that is, there exists $z^\star\in X$ such that

$$\lim_{n\to+\infty} S(y_n,y_n,z^\star)=0.$$

Consequently, we can find $u\in X$ such that $fu=z^\star$. Now to show that $Tu=z^\star$. From (4.4), we have

$$S(fx_n,fx_n,fu)\geq aS(Tx_n,Tx_n,Tu)+bS(Tx_n,Tx_n,fx_n)+cS(Tu,Tu,fu)$$

Taking limit as $n\to+\infty$ in the above inequality, we get

$$\begin{aligned}0=S(,z^\star,z^\star,fu)&\geq aS(z^\star,z^\star,Tu)+bS(z^\star,z^\star,z^\star)+\mathrm{c}S(Tu,Tu,z^\star)\\&=(a+c)S(z^\star,z^\star,Tu)\text{ or }(a+c)S(Tu,Tu,z^\star)\end{aligned}$$

This implies that $S(z^\star,z^\star,Tu)=0$ or $S(Tu,Tu,z^\star)=0$ and so $Tu=z^\star$. Therefore, $fu=Tu=z^\star$. Therefore, $z^\star$ is a point of coincidence of f and T.

Now we suppose that $a>1$. Let $w^\star$ be another point of coincidence of f and T. So $Sv=Tv=w^\star$ for some $v\in X$. Then from (4.7), we have

$$\begin{aligned}S(z^\star,z^\star,w^\star)&=S(fu,fu,fv)\\&\geq aS(Tu,Tu,Tv)+bS(Tu,Tu,fu)+cS(Tv,Tv,fv)\\&=aS(z^\star,z^\star,w^\star)\end{aligned}$$

This is true only when $S(z^\star,z^\star,w^\star)=0$. So $\mathrm{z}^\star=\mathrm{w}^\star$.

Since f and T be weakly compatible, $fTu=Tfu$, that is, $fz^\star=Tz^\star$. Now we show that $z^\star$ is a common fixed point of f and T. If $a>1$, then from condition (4.7), we have

$$S(fx_n,fx_n,fz^\star)\geq aS(Tx_n,Tx_n,Tz^\star)+bS(Tx_n,Tx_n,fx_n)+cS(Tz^\star,Tz^\star,fz^\star)$$

Proceeding to the limit as $n\to+\infty$, we have

$$S(z^\star,z^\star,fz^\star)\geq aS(z^\star,z^\star,Tz^\star)=aS(z^\star,z^\star,fz^\star),$$

which implies that $S(z^\star,z^\star,fz^\star)=0$. Hence $fz^\star=z^\star$ and so $fz^\star=Tz^\star=z^\star$. Hence f and T have a unique fixed point in X. This completes the proof.

Conflict of Interest

No conflict of interest was declared by the authors.

Author's Contributions

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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