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Predictor – Corrector Methods of High Order for Numerical Integration of Initial Value Problems

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Abstract: *Two tenth order implicit linear multistep methods are derived after applying appropriate order conditions to the Taylor series approach in the derivation of linear multistep methods. Each of the derived schemes is further combined with an Adams – Bashforth scheme of order ten to form two separate predictor – corrector pairs for numerical integration of initial value problems of ordinary differential equations. A tenth order Runge – Kutta method is further employed in order to generate the necessary starting values typical of linear multistep methods. The derived schemes are proven to be convergent by satisfying both consistency and zero – stability requirements. Numerical examples are further carried out to ascertain their efficiency and effectiveness.*

Keywords: *Predictor – corrector method, Linear multistep method, Runge – kutta method, Stability, Adams – Bashforth method.*

1. Introduction

A great many problems in science, engineering and real life applications, more often than not, are modeled as systems of first-order differential equations in n dependent variables $y_1, y_2, \dots, y_n$ thus:

$$\left.\begin{aligned} y_1' &= f_1(x_1, y_1, y_2, \dots, y_n) \\ y_2' &= f_2(x_1, y_1, y_2, \dots, y_n) \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ y_n' &= f_1(x_1, y_1, y_2, \dots, y_n) \end{aligned}\right\} \tag{1}$$

$$y_1(x_0) = y_{10}, y_2(x_0) = y_{20}, \cdots, y_n(x_0) = y_{n0} \tag{2}$$

An alternative form of writing the initial value problem (IVP) (1) and (2) is

$$\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y}), \ \ \boldsymbol{y}(x_0) = \boldsymbol{y}_0 \tag{3}$$

where, $\boldsymbol{y} = [y_1, y_2, \dots, y_n]^T$, $\boldsymbol{f} = [f_1, f_2, \dots, f_n]^T = \boldsymbol{f}(x, \boldsymbol{y})$, $\boldsymbol{y}_0 = [y_{10}, y_{20}, \dots, y_{n0}]^T$.

A numerical solution of the IVP (3) consists of a sequence of values $\{y_n\}$ which estimates the solution (3) on the distinct point set $\{x_n\}$. Suppose a $k-\text{step}$ linear multistep method (LMM) is employed to solve (3), then each step of the solution process involves the equation

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, y_{n+k}) + h\sum_{j=0}^{k-1} \beta_j f_{n+j} \tag{4}$$

where $y_{n+j}, f_{n+j}, j = 0, 1, \cdots, k-1$ are known [1]. We proceed by using a separate explicit linear multistep method to estimate an approximation to y_{n+k} and use this predicted value on the right hand side of (4) in the evaluation of $f(x_{n+k}, y_{n+k})$. The implicit method is then used for a predetermined number of steps. The explicit method is called the predictor and the implicit method is called the corrector. Popular predictor – corrector methods in use include the Milne's method [2], Hamming's method [3], Klopfenstein – Millman algorithm [4], Crane – Klopfenstein algorithm [5], Krogh's method [6] and Ndanusa and Adeboye' s method [7].

In this paper, we analyze the development of two implicit linear multistep methods of order ten which are combined with a tenth order Adams – Bashforth method, to form two predictor – corrector methods for the numerical integration of initial value problems.

2. Methods

2.1 Derivation of the LMMs of Order Ten

Since it is required that the corrector methods be of order 10, the step number of the methods, k, has to be 8. Thus, the first characteristic polynomial $\rho(\xi)$ takes the form

$$\begin{aligned}\rho(\xi) = \xi^8 - (2a+2b+2c)\xi^7 + (2+4bc+4ac+4ab)\xi^6 \\ +(-2b-2c-4a-8abc-2a)\xi^5 - (2a-4a-8abc-2c-2b)\xi^3 \\ -(-4ab-4ac-a-2bc)\xi^2 + (2a+2b+2c)\xi - 1 \qquad (5)\end{aligned}$$

where

$$\left.\begin{array}{lll} \alpha_8 = +1, & \alpha_7 = -2(a+b+c), & \alpha_6 = 2(1+2ab+2ac+2bc) \\ \alpha_5 = -2(a+b+c+4abc), & \alpha_4 = 0, & \alpha_3 = 2(a+b+c+4abc) \\ \alpha_2 = -2(1+2ab+2ac+2bc), & \alpha_1 = 2(a+b+c), & \alpha_0 = -1 \end{array}\right\} \qquad (6)$$

Therefore, the order conditions are expressed as follows

$$D_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 \qquad (7)$$

$$\begin{aligned}D_1 = [-r\alpha_0 + (1-r)\alpha_1 + (2-r)\alpha_2 + \cdots + (7-r)\alpha_7 + (8-r)\alpha_8] \\ -[\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 + \beta_7 + \beta_8] \qquad (8)\end{aligned}$$

$$\begin{aligned}D_2 = \frac{1}{2!}[(-r)^2\alpha_0 + (1-r)^2\alpha_1 + (2-r)^2\alpha_2 + \cdots + (7-r)^2\alpha_7 + (8-r)^2\alpha_8] \\ -[-r\beta_0 + (1-r)\beta_1 + (2-r)\beta_2 + \cdots + (7-r)\beta_7 + (8-r)\beta_8] \qquad (9)\end{aligned}$$

$$\begin{aligned}D_3 = \frac{1}{3!}[(-r)^3\alpha_0 + (1-r)^3\alpha_1 + (2-r)^3\alpha_2 + \cdots + (7-r)^3\alpha_7 + (8-r)^3\alpha_8] \\ -\frac{1}{2!}[-r^2\beta_0 + (1-r)^2\beta_1 + (2-r)^2\beta_2 + \cdots + (7-r)^2\beta_7 + (8-r)^2\beta_8] \qquad (10)\end{aligned}$$

$$\begin{aligned}D_4 = \frac{1}{4!}[(-r)^4\alpha_0 + (1-r)^4\alpha_1 + (2-r)^4\alpha_2 + \cdots + +(7-r)^4\alpha_7 + (8-r)^4\alpha_8] \\ -\frac{1}{3!}[-r^3\beta_0 + (1-r)^3\beta_1 + (2-r)^3\beta_2 + \cdots + +(7-r)^3\beta_7 + (8-r)^3\beta_8] \qquad (11)\end{aligned}$$

$$D_5 = \frac{1}{5!}[(-r)^5\alpha_0 + (1-r)^5\alpha_1 + (2-r)^5\alpha_2 + \cdots + (7-r)^5\alpha_7 + (8-r)^5\alpha_8]$$

$$-\frac{1}{4!}[-r^4\beta_0+(1-r)^4\beta_1+(2-r)^4\beta_2+\cdots+(7-r)^4\beta_7+(8-r)^4\beta_8] \quad (12)$$

$$D_6=\frac{1}{6!}[(-r)^6\alpha_0+(1-r)^6\alpha_1+(2-r)^6\alpha_2+\cdots+(7-r)^6\alpha_7+(8-r)^6\alpha_8]$$

$$-\frac{1}{5!}[-r^5\beta_0+(1-r)^5\beta_1+(2-r)^5\beta_2+\cdots+(7-r)^5\beta_7+(8-r)^5\beta_8] \quad (13)$$

$$D_7=\frac{1}{7!}[(-r)^7\alpha_0+(1-r)^7\alpha_1+(2-r)^7\alpha_2+\cdots+(7-r)^7\alpha_7+(8-r)^7\alpha_8]$$

$$-\frac{1}{6!}[-r^6\beta_0+(1-r)^6\beta_1+(2-r)^6\beta_2+\cdots+(7-r)^6\beta_7+(8-r)^6\beta_8] \quad (14)$$

$$D_8=\frac{1}{8!}[(-r)^8\alpha_0+(1-r)^8\alpha_1+(2-r)^8\alpha_2+\cdots+(7-r)^8\alpha_7+(8-r)^8\alpha_8]$$

$$-\frac{1}{7!}[-r^7\beta_0+(1-r)^7\beta_1+(2-r)^7\beta_2+\cdots+(7-r)^7\beta_7+(8-r)^7\beta_8] \quad (15)$$

$$D_9=\frac{1}{9!}[(-r)^9\alpha_0+(1-r)^9\alpha_1+(2-r)^9\alpha_2+\cdots+(7-r)^9\alpha_7+(8-r)^9\alpha_8]$$

$$-\frac{1}{8!}[-r^8\beta_0+(1-r)^8\beta_1+(2-r)^8\beta_2+\cdots+(7-r)^8\beta_7+(8-r)^8\beta_8] \quad (16)$$

$$D_{10}=\frac{1}{10!}[(-r)^{10}\alpha_0+(1-r)^{10}\alpha_1+(2-r)^{10}\alpha_2+\cdots+(7-r)^{10}\alpha_7+(8-r)^{10}\alpha_8]$$

$$-\frac{1}{9!}[-r^9\beta_0+(1-r)^9\beta_1+(2-r)^9\beta_2+\cdots+(7-r)^9\beta_7+(8-r)^9\beta_8] \quad (17)$$

$$D_{11}=\frac{1}{11!}[(-r)^{11}\alpha_0+(1-r)^{11}\alpha_1+(2-r)^{11}\alpha_2+\cdots++(7-r)^{11}\alpha_7+(8-r)^{11}\alpha_8]$$

$$-\frac{1}{10!}[-r^{10}\beta_0+(1-r)^{10}\beta_1+(2-r)^{10}\beta_2+\cdots++(7-r)^{10}\beta_7+(8-r)^{10}\beta_8] \quad (18)$$

By setting $r=4$ in Equations (8) to (17), $D_q=0, q=2,3,\cdots\cdots10$, and simplifying the resulting set of equations we obtain the following results.

$$\beta_0=\frac{23}{14175}abc+\frac{52}{14175}ab+\frac{52}{14175}ac+\frac{52}{14175}bc+\frac{188}{14175}a$$
$$+\frac{188}{14175}b+\frac{188}{14175}c+\frac{3982}{14175}=\beta_8 \quad (19)$$

$$\beta_1=-\frac{334}{14175}abc-\frac{128}{2025}ab-\frac{128}{2025}ac-\frac{128}{2025}bc-\frac{9844}{14175}a$$
$$-\frac{9844}{14175}b+\frac{9844}{14175}c+\frac{23104}{14175}=\beta_7 \quad (20)$$

$$\beta_2=\frac{2804}{14175}abc+\frac{21976}{14175}ab+\frac{21976}{14175}ac+\frac{21976}{14175}bc-\frac{37936}{14175}a$$
$$-\frac{37936}{14175}b-\frac{37936}{14175}c+\frac{7276}{14175}=\beta_6 \quad (21)$$

$$\beta_3 = -\frac{46378}{14175}abc + \frac{70528}{14175}ab + \frac{70528}{14175}ac + \frac{70528}{14175}bc - \frac{27268}{14175}a - \frac{27268}{14175}b - \frac{27268}{14175}c + \frac{77248}{14175} = \beta_5 \qquad (22)$$

$$\beta_4 = -[\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_5 + \beta_6 + \beta_7 + \beta_8] + 8 \\ 12(a+b+c) + 8(1 + 2ab + 2ac + 2bc) + 4(a+b+c) \qquad (23)$$

And the expression for the error constant D_{11} is

$$D_{11} = \frac{2}{11!}[-4^{11}\alpha_0 - 3^{11}\alpha_1 - 2^{11}\alpha_2 - \alpha_3] - \frac{2}{10!}[4^{10}\beta_0 + 3^{10}\beta_1 + 2^{10}\beta_2 + \beta_3] \qquad (24)$$

The choice of values for the free parameters a, b and c is inspired by the desire to reduce the error constant as well as to derive a method that makes computation easier by decreasing the number of operations involved. The following are the preferable choices for the parameters a, b and c.

$$a = \frac{1}{4}, \qquad b = \frac{1}{12}, \qquad c = -\frac{1}{3} \qquad (25)$$

and

$$a = \frac{1}{2}, \qquad b = \frac{2}{3}, \qquad c = -\frac{1}{2} \qquad (26)$$

And the corresponding LMM schemes are

$$y_{n+8} + \frac{59}{36}y_{n+6} + \frac{1}{18}y_{n+5} - \frac{1}{18}y_{n+3} - \frac{59}{36}y_{n+2} - y_n = h[\frac{190903}{680400}f^{(p)}{}_{n+8} + \frac{79499}{48600}f_{n+7} + \frac{63271}{170100}f_{n+6} + \frac{1708871}{340200}f_{n+5} + \frac{2977}{68040}f_{n+4} + \frac{1708871}{340200}f_{n+3} + \frac{63277}{170100}f_{n+2} + \frac{79499}{48600}f_{n+1} + \frac{190903}{680400}f_n] \qquad (27)$$

and

$$y_{n+8} - \frac{4}{3}y_{n+7} + y_{n+6} - y_{n+2} + \frac{4}{3}y_{n+1} - y_n = h[\frac{101}{350}f^{(p)}{}_{n+8} + \frac{89}{75}f_{n+7} - \frac{296}{175}f_{n+6} + \frac{607}{175}f_{n+5} - \frac{263}{105}f_{n+4} + \frac{607}{175}f_{n+3} - \frac{296}{175}f_{n+2} + \frac{89}{75}f_{n+1} + \frac{101}{350}f_n] \qquad (28)$$

respectively.

The error constants for the derived schemes (27) and (28) are computed to be $-\frac{4309}{816480}$ and $-\frac{139}{23100}$ respectively.

2.2 Convergence Analysis

The first characteristic polynomial $\rho(\xi)$ of scheme (27) is

$$\rho(\xi) = \xi^8 + \frac{59}{36}\xi^6 + \frac{1}{18}\xi^5 - \frac{1}{18}\xi^3 - \frac{59}{36}\xi^2 - 1 \qquad (29)$$

From the first characteristic polynomial $\rho(\xi)$ of the scheme (27) we have

$$\rho(1) = \sum_{j=0}^{8} \alpha_j = 1 + \frac{59}{36} + \frac{1}{18} - \frac{1}{18} - \frac{59}{36} - 1 = 0 \qquad (30)$$

$$\rho'(1) = \sum_{j=0}^{8} j\alpha_j = 8(1) + \frac{59}{6}(1) + \frac{5}{18}(1) - \frac{1}{6}(1) - \frac{59}{18} = \frac{44}{3} \qquad (31)$$

And the second characteristic polynomial $\sigma(\xi)$ of scheme (27) is

$$\sigma(\xi) = \frac{190903}{680400}\xi^8 + \frac{79499}{48600}\xi^7 + \frac{63271}{170100}\xi^6 + \frac{1708871}{340200}\xi^5 + \frac{2977}{68040}\xi^4 + \frac{1708871}{340200}\xi^3 + \frac{63277}{170100}\xi^2 + \frac{79499}{48600}\xi + \frac{190903}{680400} \qquad (32)$$

From Equation (32) we compute

$$\sigma(1) = \frac{44}{3} \qquad (32a)$$

From Equations (30), (31) and (32a) we observed that

$$\left.\begin{array}{ll} (i) & \rho(1) = 0 \\ (ii) & \rho'(1) = \sigma(1) \end{array}\right\} \qquad (33)$$

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Therefore the scheme (27) is consistent.

Also, the roots of Equation (29) are computed as follows

$$\left.\begin{array}{l} \xi_1 = 1, \quad \xi_2 = -1, \; \xi_3 = \frac{1}{4} - \frac{1}{4}\sqrt{15}\, i, \quad \xi_4 = \frac{1}{4} + \frac{1}{4}\sqrt{15}\, i \\ \xi_5 = -\frac{1}{3} - \frac{2}{3}\sqrt{2}\, i, \quad \xi_6 = -\frac{1}{3} + \frac{2}{3}\sqrt{2}\, i \\ \xi_7 = \frac{1}{12} - \frac{1}{12}\sqrt{143}\, i, \quad \xi_8 = \frac{1}{12} + \frac{1}{12}\sqrt{143}\, i \end{array}\right\} \qquad (34)$$

Each of the roots in Equations (34) have a modulus of 1. Thus the scheme (27) is also zero – stable. Hence, it is convergent.

Similarly, the first characteristic polynomial $\rho(\xi)$ of scheme (28) is

$$\rho(\xi) = \xi^8 - \frac{4}{3}\xi^7 + \xi^6 - \xi^2 + \frac{4}{3}\xi - 1 \qquad (35)$$

From Equation (35) we have

$$\rho(1) = 1 - \frac{4}{3} + 1 - 1 + \frac{4}{3} - 1 = 0 \qquad (36)$$

and

$$\rho'(\xi) = 8\xi^7 - \frac{28}{3}\xi^6 + 6\xi^5 - 2\xi + \frac{4}{3} = 8 - \frac{28}{3} + 6 - 2 + \frac{4}{3} = 4 \qquad (37)$$

And the second characteristic polynomial $\sigma(\xi)$ of scheme (28) is

$$\sigma(\xi) = \frac{101}{350}\xi^8 + \frac{89}{75}\xi^7 - \frac{296}{175}\xi^6 + \frac{607}{175}\xi^5 - \frac{263}{105}\xi^4 + \frac{607}{175}\xi^3 - \frac{296}{175}\xi^2 + \frac{89}{75}\xi + \frac{101}{350} \qquad (38)$$

From Equation (38) we compute

$$\sigma(1) = 4 \tag{39}$$

From Equations (36), (37) and (39) we observed that

$$\left.\begin{array}{ll} (i) & \rho(1) = 0 \\ (ii) & \rho'(1) = \sigma(1) \end{array}\right\} \tag{40}$$

Therefore the scheme (28) is consistent.

Also, the roots of Equation (35) are computed as follows

$$\left.\begin{array}{l} \xi_1 = 1, \quad \xi_2 = -1, \; \xi_3 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}\, i, \quad \xi_4 = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\, i \\ \xi_5 = \frac{2}{3} - \frac{1}{3}\sqrt{5}\, i, \quad \xi_6 = \frac{2}{3} + \frac{1}{3}\sqrt{5}\, i \\ \xi_7 = \frac{1}{2} - \frac{1}{2}\sqrt{3}\, i, \quad \xi_8 = \frac{1}{2} + \frac{1}{2}\sqrt{3}\, i \end{array}\right\} \tag{41}$$

Each of the roots in Equations (41) have a modulus of 1. Thus the scheme (28) is also zero – stable. Hence, it is convergent.

2.3 Numerical Experiments

The following scheme is an explicit tenth – order Adams –Bashforth method, due to [8].

$$\begin{aligned} y_{n+1} = y_n + \frac{h}{7257600}[&49537553f_n - 259077637f_{n-1} + 805221248f_{n-2} \\ &-1533238912f_{n-3} + 1886585258f_{n-4} - 1523349298f_{n-5} + 791906792f_{n-6} \\ &-248389768f_{n-7} + 401445117f_{n-8} - 2082753f_{n-9}] \end{aligned} \tag{42}$$

Scheme (42) is used as a predictor for schemes (27) and (28) to form two sets of predictor – corrector pairs for the numerical solution of initial value problems. In order to generate the starting values for the predictor – corrector schemes, a tenth –order Runge – Kutta method due to Hairer [9] is employed.

The following problems are solved with the derived methods.

Problem 1: $y' = x + y, \; y(0) = 1, \; h = 0.1, \; 0 \le x \le 2$

Exact Solution: $Y_E(x) = 2e^x - 1 - x$

Problem 2: $y' = 7x^6 - 10x^4 + 9x^2 + 2, \; y(0) = 1, \; h = 0.1, \; 0 \le x \le 2$

Exact Solution: $Y_E(x) = x^7 - 2x^5 + 3x^3 + 2x + 1$

3. Results and Discussion

A Tables 1 to 4 depict the results of applyin the predictor – corrector methods to some sample problems.

Table 1. *Results of problem 1 with scheme (27)*

x	Exact solution	Approximate	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.1103418362	1.1103418362	4.8759885019E-11
0.2	1.2428055163	1.2428055163	2.0210055851E-11
0.3	1.3997176152	1.3997176152	4.8219872539E-11
0.4	1.5836493953	1.5836493953	1.7799983709E-11
0.5	1.7974425414	1.7974425414	2.2981616610E-13
0.6	2.0442376008	2.0442376008	1.9649615268E-11

0.7	2.3275054149	2.3275054149	4.0069725316E-11
0.8	2.6510818570	2.6510818570	1.6209700249E-11
0.9	3.0192062223	3.0192062223	1.2440271036E-11
1.0	3.4365636569	3.4365636572	2.9999958073E-10
1.1	3.9083320479	3.9083320482	3.0000002482E-10
1.2	4.4402338455	4.4402338454	1.0000000827E-10
1.3	5.0385933352	5.0385933352	0.0000000000E+00
1.4	5.7103999337	5.7103999342	5.0000004137E-10
1.5	6.4633781407	6.4633781414	7.0000005792E-10
1.6	7.3060648488	7.3060648491	3.0000002482E-10
1.7	8.2478947835	8.2478947837	2.0000001655E-10
1.8	9.2992949288	9.2992949293	5.0000004137E-10
1.9	10.4717888846	10.4717888856	1.0000000827E-09
2.0	11.7781121979	11.7781121993	1.4000001158E-09

Table 2. *Results of problem 1 with scheme (28)*

x	Exact solution	Approximate	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.1103418362	1.1103418362	4.8759885019E-11
0.2	1.2428055163	1.2428055163	2.0210055851E-11
0.3	1.3997176152	1.3997176152	4.8219872539E-11
0.4	1.5836493953	1.5836493953	1.7799983709E-11
0.5	1.7974425414	1.7974425414	2.2981616610E-13
0.6	2.0442376008	2.0442376008	1.9649615268E-11
0.7	2.3275054149	2.3275054149	4.0069725316E-11
0.8	2.6510818570	2.6510818570	1.6209700249E-11
0.9	3.0192062223	3.0192062223	1.2440271036E-11
1.0	3.4365636569	3.4365636572	2.9999958073E-10
1.1	3.9083320479	3.9083320486	7.0000005792E-10
1.2	4.4402338455	4.4402338465	1.0000000827E-09
1.3	5.0385933352	5.0385933364	1.2000000993E-09
1.4	5.7103999337	5.7103999348	1.1000000910E-09
1.5	6.4633781407	6.4633781416	9.0000007447E-10
1.6	7.3060648488	7.3060648500	1.2000000993E-09
1.7	8.2478947835	8.2478947853	1.8000001489E-09
1.8	9.2992949288	9.2992949315	2.7000002234E-09
1.9	10.4717888846	10.4717888879	3.3000002730E-09
2.0	11.7781121979	11.7781122016	3.7000003061E-09

Table 3. *Results of problem 2 with scheme (27)*

x	Exact solution	Approximate	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.2029801000	1.2029801000	0.0000000000E+00
0.2	1.4233728000	1.4233728000	0.0000000000E+00
0.3	1.6763587000	1.6763587000	0.0000000000E+00
0.4	1.9731584000	1.9731584000	0.0000000000E+00
0.5	2.3203125000	2.3203125000	0.0000000000E+00
0.6	2.7204736000	2.7204736000	0.0000000000E+00
0.7	3.1752143000	3.1752143000	0.0000000000E+00
0.8	3.6903552000	3.6903552000	0.0000000000E+00

0.9	4.2843169000	4.2843169000	0.0000000000E+00
1.0	5.0000000000	5.0000000000	0.0000000000E+00
1.1	5.9206971000	5.9206971000	0.0000000000E+00
1.2	7.1905408000	7.1905408000	0.0000000000E+00
1.3	9.0399917000	9.0399917000	0.0000000000E+00
1.4	11.8168704000	11.8168704000	0.0000000000E+00
1.5	16.0234375000	16.0234375000	0.0000000000E+00
1.6	22.3600256000	22.3600256000	0.0000000000E+00
1.7	31.7757273000	31.7757273000	0.0000000000E+00
1.8	45.5266432000	45.5266432000	0.0000000000E+00
1.9	65.2421939000	65.2421939000	0.0000000000E+00
2.0	93.0000000000	93.0000000000	0.0000000000E+00

Table 4. *Results of problem 2 with scheme (28)*

x	Exact solution	Approximate	Error
0.0	1.0000000000	1.0000000000	0.0000000000E+00
0.1	1.2029801000	1.2029801000	0.0000000000E+00
0.2	1.4233728000	1.4233728000	0.0000000000E+00
0.3	1.6763587000	1.6763587000	0.0000000000E+00
0.4	1.9731584000	1.9731584000	0.0000000000E+00
0.5	2.3203125000	2.3203125000	0.0000000000E+00
0.6	2.7204736000	2.7204736000	0.0000000000E+00
0.7	3.1752143000	3.1752143000	0.0000000000E+00
0.8	3.6903552000	3.6903552000	0.0000000000E+00
0.9	4.2843169000	4.2843169000	0.0000000000E+00
1.0	5.0000000000	5.0000000000	0.0000000000E+00
1.1	5.9206971000	5.9206971000	0.0000000000E+00
1.2	7.1905408000	7.1905408000	0.0000000000E+00
1.3	9.0399917000	9.0399917000	0.0000000000E+00
1.4	11.8168704000	11.8168704000	0.0000000000E+00
1.5	16.0234375000	16.0234375000	0.0000000000E+00
1.6	22.3600256000	22.3600256000	0.0000000000E+00
1.7	31.7757273000	31.7757273000	0.0000000000E+00
1.8	45.5266432000	45.5266432000	0.0000000000E+00
1.9	65.2421939000	65.2421939000	0.0000000000E+00
2.0	93.0000000000	93.0000000000	0.0000000000E+00

In Table 1, Scheme (27) exhibited a steady rise in errors as the step increases for the first few steps i.e., from $x = 0.0$ to $x = 0.4$; and at $x = 0.5$ it has $2.2981616610E-13$, while it goes back to the same trend from $x = 0.6$ to $x = 0.9$. Thereafter, from $x = 1.0$ to $x = 1.2$ it shows a steady growth. At $x = 1.3$, it has $0.0000000000E+00$. And from $x = 1.4$ to $x = 1.8$ it maintains the same trend as in $= 1.0$ to $x = 1.2$. Then it decreases between $x = 1.9$ and $x = 2.0$. Generally, the scheme exhibited minimal error.

In Table 2, Scheme (28) demonstrated a similar trend as in Table 1. At $x = 0.5$ the error is $2.2981616610E-13$; while at $x = 1.0$ the error is $2.9999958073E-10$. Finally at $x = 2.0$ the error stands at $3.7000003061E-09$.

In Tables 3 and 4, the two schemes solved the differential equation exactly, with no error at all. This is expected, as the exact solution of problem 2 is a polynomial of degree 6, i.e., its degree is less than 8, which is the step length of the methods.

4. Conclusion

Two implicit LMMs of order ten are derived and combined with an explicit method of the same order to form predictor – corrector pairs. The derived schemes have proven to be effective and efficient by satisfying convergence criteria as well as solving some sample differential equation problems correctly.

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