New Insight of Discrete-Time Food Chain Interaction Model

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Abstract: *The discrete- time food chain interaction model is proposed. Steady state implies all possible equilibrium points. Stability conditions of arising equilibrium points are analyzed with numerical examples. Further, the dynamical behavior of the coexistence equilibrium point is obtained.*

Keywords: *discrete- time model, food chain, Stability theory, equilibrium point.*

1. INTRODUCTION

Linear Lotka-Volterra model [1] consists of two interacting species vis. Prey and predator. Such mathematical models have long proved useful in describing how populations vary over time [2]. The other basic sequence of energy is the food chain which describes the movement from producer to consumer then to decomposer which called lowest-level, mid-level and top-level, respectively. The chain food is more important in understanding the food and energy relationship in the ecosystems.

The continuous time food chain of three or more species arise in (Hastings et al.1991, McCann et al.1994, Kuznetsov et al.1996, El-Owaidy 2001, and Chauvet 2002) while the Lotka-Volterra discrete-time food chain model proposed in (Abd-Elalim et al. 2012, Hari 2014, and Sohel 2015).

In [10] we studied a non-linear discrete-time prey-predator model such that each population cans model by the logistic equation. For more insight and covering three species ecosystems, we extend our model in [10] by adding third population. The new model includes the lowest-level prey X_n is preyed upon by a mid-level species Y_n, which, in turn, preyed upon by a top-level predator Z_n as shown in the following system:

$$X_{n+1} = X_n + h[aX_n(1 - X_n) - bX_nY_n]$$

$$Y_{n+1} = Y_n + h[cY_n(1 - Y_n) + bX_nY_n - fY_nZ_n] \qquad (1)$$

$$Z_{n+1} = Z_n + h(-d + fY_n)Z_n$$

Where a, b, c, d and f are all positive parameters.

2. STEADY STATE

In common parlance the system (1) has steady state if it is equal to the vector X^T where $X = \{X, Y, Z\}$ .Thus, this implies the following equilibrium points:

i.   Fully washed out state

- $(x, y, z) = (0, 0, 0)$

ii- States in which only one of the three species is survives while the other two are not

- $(x, y, z) = (1, 0, 0)$

- $(x, y, z) = (0, 1, 0)$

iii- States in which only two of the three species are survives while the other one is not

- $(x, y, z) = (\frac{ac-bc}{ac+b^2}, \frac{ac+ab}{ac+b^2}, 0)$ where $a > b$.

- $(x, y, z) = (0, \frac{d}{f}, \frac{cf-cd}{f^2})$ where $f > d$.

iv- The co-existent steady states

$(x, y, z) = (\frac{af-bd}{af}, \frac{d}{f}, \frac{ac(f-d)+b(af-bd)}{af^2})$ where $a \geq b$ and $f > d$.

3. STABILITY AND NUMERICAL ANALYSIS

In general, if we want to study the stability of an equilibrium point in 3 dimensional systems we should firstly compute the Jacobian matrix and its characteristic equation then find out their roots as follows:

$$J(x, y, z) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{2}$$

$$F(\lambda) = \lambda^3 + M_1\lambda^2 + M_2\lambda + M_3 = 0 \tag{3}$$

Where

$M_1 = -(a_{11} + a_{22} + a_{33})$

$M_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{23}a_{32} - a_{12}a_{21} - a_{13}a_{31}$

$M_3 = a_{11}(a_{23}a_{32} - a_{22}a_{33}) + a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{13}a_{22} - a_{12}a_{23})$

Remark: The equilibrium point (x,y,z) is stable if and only if all the absolute values of the roots of the equation (3) are less than one[10].

3.1 Proposition

The equilibrium points $(0,0,0)$ and $(1,0,0)$ in the states (i) and (ii), resp., are unstable.

3.2 Proposition

The equilibrium point $(0, 1, 0)$ is stable if $h < min\{\frac{2}{b-a}, \frac{2}{c}, \frac{2}{d-f}\}$

3.2.1 Example

In the system (1) we have that the following parameters values

$a = 0.12, b = 0.15, c = 0.1, d = 0.95$ and $f = 0.04$ where inertial point

$(x_0, y_0, z_0) = (0.5, 0.3, 0.4)$ and $h = 0.1$. Under the above fixed values we see that the equilibrium point $(0,1,0)$ is stable. See the phase portrait and time series in the figure (1) bellow.
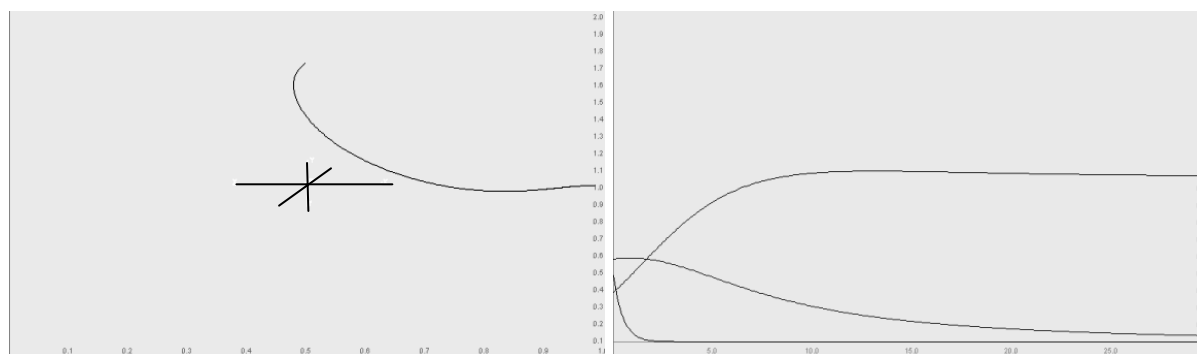

Figure (1). *Stability of the equilibrium point* $(0,1,0)$.

3.3 Proposition

Let $a > b$ then we have that the equilibrium point $(\frac{ac-bc}{ac+b^2}, \frac{ac+ab}{ac+b^2}, 0)$ is stable if the following conditions hold:

1. $d - 2 < h < d$.

2. $\left| \frac{-T \pm \sqrt{T^2 - 4D}}{2} \right| < 1$

where

$$T = -(2 + h\frac{ac^2 - a^2 c}{ac+b^2}) \text{ and } D = \frac{ac(1+hc-ha)+b^2}{ac+b^2} + h^2 \frac{a^2 b^3 c - ab^3 c^2 - ab^4 c + abc^3 + a^3 bc^2 - a^3 c^3}{ac+b^2}.$$

With the absence of top-level species, system (1) can be reduced to a sub system as discrete –time prey-predator which we studied before. This sub model exhibit complex dynamic behavior. For more details see [10].

3.3.1 Example

In the system (1) we have that the following parameters values $a = 0.7, b = 0.65, c = 0.1, d = 0.5$ and $f = 1.14$ where inertial point $(x_0, y_0, z_0) = (0.5, 0.3, 0.4)$ and $h = 0.1$. Under the above fixed values we see that the equilibrium point $(0.01015, 1.06598, 0)$ is stable. See the phase portrait and time series in the figure (2).
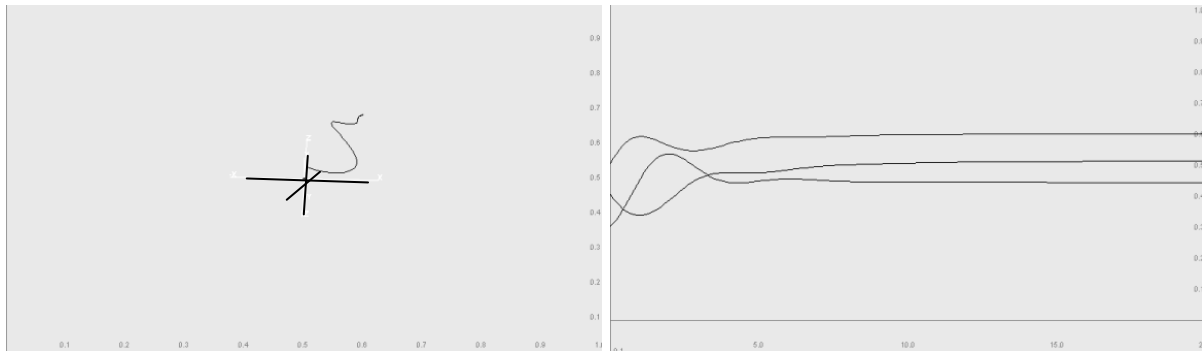

Figure (2). *Stability of the equilibrium point* $(0.01015, 1.06598, 0)$.

3.4 Proposition

If $f > d$ then the equilibrium point $(0, \frac{d}{f}, \frac{cf-cd}{f^2})$ is stable if the following conditions hold:

1. $h < \frac{2f}{bd-af}$.

2. $\left| \frac{\frac{2f-hcd}{f} \pm \sqrt{(\frac{2f-hcd}{f})^2 - \frac{4(f-hcd(f-d))}{f}}}{2} \right| < 1$

With the absence of lowest-level system (1) can be reduced to a sub system as discrete –time prey-predator which studied by Liu and Xiao. This sub model has results reveal far richer dynamics of the discrete model compared with the continuous model. For more details see [12]

3.4.1 Example

In the system (1) we have that the following parameters values $a = 0.1, b = 0.65, c = 5, d = 0.5$ and $f = 1.14$ where inertial point $(x_0, y_0, z_0) = (0.5, 0.3, 0.4)$ and $h = 0.7$. Under the above fixed values we see that the equilibrium point $(0, 0.4389, 2.46229)$ is stable. See the bellow phase portrait and time series in the figure (3).
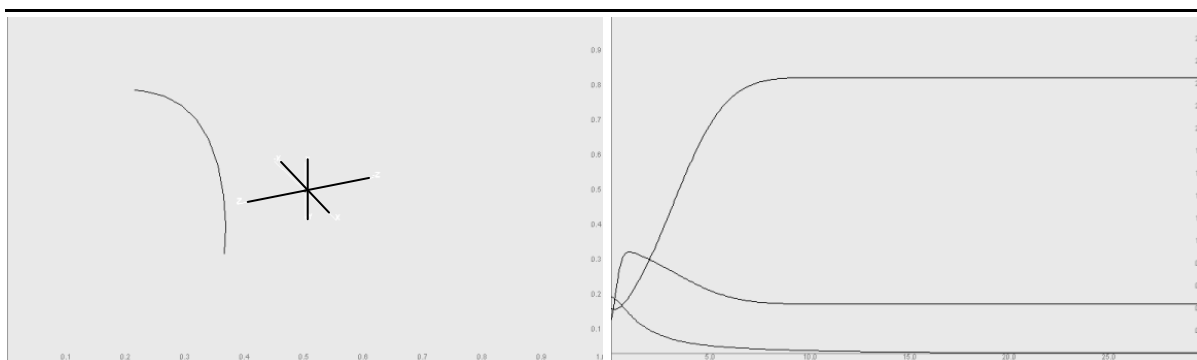
Figure (3). *Stability of the equilibrium point* $(0, 0.4389, 2.46229)$.

3.5 Proposition

If $a \geq b$ and $f > d$ then the coexisting equilibrium point $(\frac{af-bd}{af}, \frac{d}{f}, \frac{ac(f-d)+b(af-bd)}{af^2})$ is locally asymptotically stable if all the following Routh-Hurwitz conditions[11] hold:

1. $3 + M_1 - M_2 - 2M_3 > 0$;

2. $1 - M_2 + M_3(M_1 - M_3) > 0$;

3. $1 - M_1 + M_2 - M_3 > 0$.

Where

$$M_1 = -(3 + h\frac{f(1+c-a)+d(1+b-2c)}{f}),$$

$$M_2 = \frac{3f+2h(bd+cf+d-af-2cd-f)}{f} + h^2 \frac{a(bd-af)(d+cf-2cd-f)-(bd-af)(b^2d+bdf)+acdf(f-d)}{af^2} \quad \text{and}$$

$$M_3 = \frac{f+h(cf+d-f-2cd-af+bd)}{f} + h^2 d \frac{acf(f-d)+b^2(af-bd)}{af^2} + h^3 d \frac{(bd-af)[ac(f-d)+b(af-bd)-f]}{af^2}$$

3.5.1 Example

In the figure (4), if the parameters value as follow $a = 2.65, b = 0.5, c = 0.1, d = 0.8$ and $f = 0.9$ where inertial point $(x_0, y_0, z_0) = (0.5, 0.5, 0.5)$ and $h = 0.87$ then the coexisting equilibrium point $(0.674151, 0.888889, 0.474726)$ in the system (1) is stable Under the certain conditions. See the figure below:
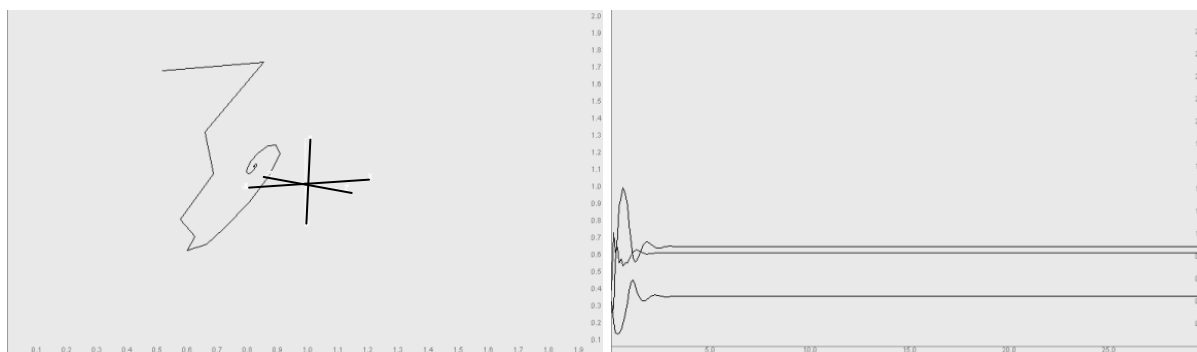

Figure (4). *Stability of the equilibrium point (0.674151, 0.888889, 0.474726).*

3 CONCLUSION

In our model, the lowest-level and mid-level are growth logistically with absence of other species. Also, if $a \geq b$ and $f > d$ then all possible equilibrium points are existing. With absence of the top-level then the control parameter h lays on the interval $[d-2, d]$ as a stability condition of the equilibrium point $(\frac{ac-bc}{ac+b^2}, \frac{ac+ab}{ac+b^2}, 0)$.

The model of discrete-time food chain makes an interesting system in mathematical modeling as well as ecosystems science. It can be modified to model more complicated biological behavior. Finally, the phaser program is used to analysis our model numerically.

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