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Expansion Formula for the Multivariable A-Function Involving Generalized Legendre's Associated Function

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Abstract: *The authors have established a new expansion formula for multivariable A-function due to Gautam et. al. [3] in terms of products of the multivariable A-function and the generalized Legendre's associated function due to Meulenbeld [4]. Some special cases are given in the last.*

Keywords: *Multivariable A-function, Generalized Legendre's associated function, Multivariable H-function.*

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1. Introduction

Gautam and Goyal [3] defined and represented the multivariable A-function as follows:

$$A[z_1,...,z_r] = A^{m,n:m_1,n_1;...;m_r,n_r}_{p,q:p_1,q_1;...;p_r,q_r}$$

$$\cdot\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left| \begin{matrix} (a_j;A'_j,...,A^{(r)}_j)_{1,p};(c'_j,C'_j)_{1,p_1};...;(c^{(r)}_j,C^{(r)}_j)_{1,p_r} \\ (b_j;B'_j,...,B^{(r)}_j)_{1,q};(d'_j,D'_j)_{1,q_1};...;(d^{(r)}_j,D^{(r)}_j)_{1,q_r} \end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}....\int_{L_r}\theta_1(s_1)....\theta_r(s_r)\Phi(s_1,....,s_r)z_1^{s_1}...z_r^{s_r}\ .ds_1...ds_r \tag{1.1}$$

Where $\omega = \sqrt{-1}$;

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma(d^{(i)}_j - D^{(i)}_j s_i)\prod_{j=1}^{n_i}\Gamma(1-c^{(i)}_j + C^{(i)}_j s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d^{(i)}_j + D^{(i)}_j s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c^{(i)}_j - C^{(i)}_j s_i)}$$

$$\forall i \in \{1,\ ....,\mathrm{r}\} \tag{1.2}$$

$$\Phi(s_1,...,s_r) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}A^{(i)}_j s_i)\prod_{j=1}^{m}\Gamma(b_j-\sum_{i=1}^{r}B^{(i)}_j s_i)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}A^{(i)}_j s_i)\prod_{j=m+1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}B^{(i)}_j s_i)} \tag{1.3}$$

Here $\mathrm{m}, \mathrm{n}, \mathrm{p}, \mathrm{q}, \mathrm{m}_i, \mathrm{n}_i, \mathrm{p}_i$, and q_i $\mathrm{i} = 1,\ ....,\mathrm{r}$ are non-negative integers and all $a'_j s, b'_j s, d^{(i),}_j s, c^{(i),}_j s, A^{(i),}_j s$ and $B^{(i),}_j s$ are complex numbers.

The multiple integral defining the A-function of r-variables converges absolutely if

$$\left|\arg(\Omega_i) z_k\right| < \frac{\pi}{2}\eta_i, \xi_i^* = 0, \eta_i > 0 \tag{1.4}$$

$$\Omega_i = \prod_{j=1}^{p}\{A_j^{(i)}\}^{A_j^{(i)}} \prod_{j=1}^{q}\{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}} . \prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}}$$

$$, \forall i \in \{1, \ldots, \mathrm{r}\} ; \tag{1.5}$$

$$\xi^*_i = I_m\left[\sum_{j=1}^{p} A_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{q_i} C_j^{(i)}\right], \ \forall i \in \{1, \ldots, \mathrm{r}\} \tag{1.6}$$

$$\eta_i = \mathrm{Re}\left[\sum_{j=1}^{n} A_j^{(i)} - \sum_{j=n+1}^{p} A_j^{(i)} + \sum_{j=1}^{m} B_j^{(i)} - \sum_{j=m+1}^{q} B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\right]$$

$$\forall i \in \{1, \ldots, \mathrm{r}\} ; \tag{1.7}$$

If we take $A_j^{'}s, B_j^{'}s, C_j^{'}s$ and $D_j^{'}s$ as real and positive and $\mathrm{m} = 0$, the A-function reduces to multivariable H-function of Srivastava and Panda [7]

In this paper we will evaluate an integral involving generalized associated Legendre's function and the multivariable A-function due to Gautam [3] and apply it in deriving an expansion for the multivariable A-function in series of products of associated Legendre's function and the multivariable A-function.

2. The Integral

The integral to be evaluated is:

$$\int_{-1}^{1}(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}} P_{k-\frac{u-v}{2}}^{u,v}(x)$$

$$\times A\left[(1-x)^{\alpha_1}(1+x)^{\beta_1} z_1, \ldots, (1-x)^{\alpha_r}(1+x)^{\beta_r} z_r\right] dx$$

$$= 2^{\rho-u+v+\sigma+1}\sum_{t=0}^{\infty}\frac{(-k)_t(v-u+k+1)_t}{\Gamma(1-u+t)\ t!}\ A^{m,n+2:(m',n');\ldots; m^{(r)},n^{(r)}}_{p+2,q+1:(p',q');\ldots; p^{(r)},q^{(r)}}$$

$$\left[\begin{matrix} 2^{\alpha_1+\beta_1} z_1 \\ \vdots \\ 2^{\alpha_r+\beta_r} z_r \end{matrix}\left|\begin{matrix}(-\sigma-v;\beta_1,\ldots,\beta_r), \\ \\ (b_{rj},\beta_{rj}^{'},\ldots,\beta_{rj}^{(r)})_{1,q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(u-\rho-t;\alpha_1,\ldots,\alpha_r),(\alpha_{rj},\alpha_{rj}^{'},\ldots,\alpha_{rj}^{(r)})_{1,p}:(a_j^{'},\alpha_j^{'})_{1,p'}:\ldots:(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}} \\ (u-v-\rho-\sigma-t-1;\alpha_1+\beta_1,\ldots,\alpha_r+\beta_r),:(b_j^{'},\beta_j^{'})_{1,q'}:\ldots:(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}\end{matrix}\right] \tag{2.1}$$

The integral (2.1) is valid under the following set of conditions:

(i) $\alpha_i, \beta_i > 0; \forall i \in 1,2,\ldots,r \ ; k-\frac{u-v}{2}$ is a positive integer, k is a integer ≥ 0.

(ii) $\mathrm{Re}\left(\rho-u+\sum_{i=1}^{r}\alpha_i\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1; \mathrm{Re}\left(\sigma+v+\sum_{i=1}^{r}\beta_i\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1; (j=1,2,\ldots,m_i; i=1,2,\ldots,r)$

And the conditions given in (1.4) to (1.7) are also satisfied.

Proof: On expressing the multivariable A-function in the integrand as a multiple Mellin-Barnes type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral

$$= (2\pi w)^r \int_{L_1} \ldots \int_{L_r} \psi(s_1, \ldots, s_r) \sum_{i=1}^{r} \phi_i(s_i) z_i^{\xi_i}$$

$$\times \left\{ \int_{-1}^{1} (1-x)^{\rho - \frac{u}{2} + \sum_{i=1}^{r} \alpha_i \xi_i} (1+x)^{\sigma + \frac{v}{2} + \sum_{i=1}^{r} \beta_i \xi_i} \right.$$

$$\left. P^{u,v}_{k-\frac{u-v}{2}}(x) dx \right\} d\xi_1 \ldots d\xi_r$$

On evaluating the x-integral with the help of the integral ([5],p.343, eq. (38)):

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} P^{m,n}_{k-\frac{m-n}{2}}(x) dx$$

$$= \frac{2^{\rho+\sigma-\frac{m-n}{2}} \Gamma\left(\rho - \frac{m}{2} + 1\right) \Gamma\left(\sigma + \frac{n}{2} + 1\right)}{\Gamma\ 1-m\ \Gamma\left(\rho + \sigma - \frac{m-n}{2} + 2\right)}$$

$$\times {}_3F_2\left(-k, n-m+k+1, \rho - \frac{m}{2} + 1; 1-m, \rho - \sigma - \frac{m-n}{2} + 2; 1\right) \tag{2.2}$$

Provided that $\text{Re}\left(\rho - \frac{m}{2}\right) > -1; \text{Re}\left(\sigma + \frac{n}{2}\right) > -1$ and interpreting the result with the help of (1.1), the integral (2.1) is established.

3. EXPANSION THEOREM

Let the following conditions be established:

(i) $\beta_1, \ldots, \beta_r > 0; \alpha_1, \ldots, \alpha_r \geq 0 (or \beta_1, \ldots, \beta_r \geq 0; \alpha_1, \ldots, \alpha_r > 0)$;

(ii) $m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i = 1, \ldots, r)$ are non-negative integers where

$0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q^k \geq 0, 0 \leq n \leq p$ and the conditions given by (1.4) to (1.7) are also satisfied.

(iii) $\text{Re}(u) > -1, \text{Re}(v) > -1, \text{Re}\left(\rho - u + \sum_{i=1}^{r} \alpha_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1;$

$\text{Re}\left(\sigma + v + \sum_{i=1}^{r} \beta_i \frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > -1; (j = 1, 2, \ldots, m_i; i = 1, 2, \ldots, r)$.

Then the following expansion formula holds:

$$(1-x)^{\rho - \frac{u}{2}} (1+x)^{\sigma + \frac{v}{2}} A\left[(1-x)^{\alpha_1} (1+x)^{\beta_1} z_1, \ldots, (1-x)^{\alpha_r} (1+x)^{\beta_r} z_r\right]$$

$$= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_\mu}{N!\mu!\Gamma(1+v+N)\Gamma(1-u+\mu)}$$

$$P^{u,v}_{N-\frac{u-v}{2}}(x)A^{m,n+2:(m',n');...;m^{(r)},n^{(r)}}_{p+2,q+1:(p',q');...;p^{(r)},q^{(r)}}\left[\begin{matrix}2^{\alpha_1+\beta_1}z_1\\ \vdots\\ 2^{\alpha_r+\beta_r}z_r\end{matrix}\left|\begin{matrix}(-\sigma-v;\beta_1,...,\beta_r),\\ \\ (b_{rj},\beta_{rj}^{'},...,\beta_{rj}^{(r)})_{1,q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(u-\rho-\mu;\alpha_1,...,\alpha_r),(\alpha_{rj},\alpha_{rj}^{'},...,\alpha_{rj}^{(r)})_{1,p_r}:(a_j^{'},\alpha_j^{'})_{1,p'};...;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\\ (u-v-\rho-\sigma-\mu-1;\alpha_1+\beta_1,...,\alpha_r+\beta_r),:(b_j^{'},\beta_j^{'})_{1,q'};...;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}\end{matrix}\right] \tag{3.1}$$

Proof: Let

$$f(x)=(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}A\left[(1-x)^{\alpha_1}(1+x)^{\beta_1}z_1,...,(1-x)^{\alpha_r}(1+x)^{\beta_r}z_r\right]$$

$$=\sum_{N=0}^{\infty}C_N P^{u,v}_{N-\frac{u-v}{2}}(x) \tag{3.2}$$

Equation (3.2) is valid since $f(x)$ is continuous and of bounded variation in the interval (-1,1).

Now, multiplying both the sides of (3.2) by $P^{u,v}_{N-\frac{u-v}{2}}(x)$ and integrating with respect to x from -1 to 1; evaluating the L.H.S. with the help of (2.1) and on the R.H.S. interchanging the order of summation, using ([2],p.176,eq. (75)) and then applying orthogonality property of the generalized Legendre's associated functions ([5],p.340,eq.(27)):

$$\int_{-1}^{1}P^{u,v}_{k-\frac{u-v}{2}}(x)P^{u,v}_{N-\frac{u-v}{2}}(x)dx$$

$$=\begin{cases}0;\ if\ \ k\neq N\\ \dfrac{2^{u-v+1}k!\Gamma(k+v+1)}{(2k-u+v+1)\Gamma(k-u+1)\Gamma(k-u+v+1)};if\quad k=N\end{cases} \tag{3.3}$$

Provided that $\mathrm{Re}(u),1,\mathrm{Re}(v)<1$; we obtain

$$C_k=\frac{2^{\rho+\sigma}(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)}\sum_{\mu=0}^{k}\frac{(-k)_\mu\Gamma(k-u+v+\mu+1)}{\mu!\Gamma(k-u+\mu)}$$

$$I^{m,n+2:(m',n');...;m^{(r)},n^{(r)}}_{p+2,q+1:(p',q');...;p^{(r)},q^{(r)}}\left[\begin{matrix}2^{\alpha_1+\beta_1}z_1\\ \vdots\\ 2^{\alpha_r+\beta_r}z_r\end{matrix}\left|\begin{matrix}(-\sigma-v;\beta_1,...,\beta_r),\\ \\ (b_{rj},\beta_{rj}^{'},...,\beta_{rj}^{(r)})_{1,q},\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(u-\rho-\mu;\alpha_1,...,\alpha_r),(\alpha_{rj},\alpha_{rj}^{'},...,\alpha_{rj}^{(r)})_{1,p}:(a_j^{'},\alpha_j^{'})_{1,p'};...;(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\\ (u-v-\rho-\sigma-\mu-1;\alpha_1+\beta_1,...,\alpha_r+\beta_r),:(b_j^{'},\beta_j^{'})_{1,q'};...;(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}\end{matrix}\right] \tag{3.4}$$

Now on substituting the values of C_k in (3.2), the result follows.

4. SPECIAL CASES

If in (2.1), we put $m=0$, the multivariable A-function occurring in the left-hand side of these formulae would reduce immediately to multivariable H-function due to Srivastava et. al.[7] and we get result given by Saxena and Ramawat [6]

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}H\left[(1-x)^{\alpha_1}(1+x)^{\beta_1}z_1,...,(1-x)^{\alpha_r}(1+x)^{\beta_r}z_r\right]$$

$$=2^{\rho+\sigma}\sum_{N=0}^{\infty}\sum_{\mu=0}^{N}\frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_\mu}{N!\mu!\Gamma(1+v+N)\Gamma(1-u+\mu)}$$

$$P^{u,v}_{N-\frac{u-v}{2}}(x)H^{0,n+2:(m',n');...;m^{(r)},n^{(r)}}_{p+2,q+1:(p',q');...;p^{(r)},q^{(r)}}$$

$$\left[\begin{matrix} 2^{\alpha_1+\beta_1} z_1 \\ \vdots \\ 2^{\alpha_r+\beta_r} z_r \end{matrix}\right| \begin{matrix} (-\sigma-v;\beta_1,...,\beta_r), \\ \\ (b_j,\beta_j^{'},...,\beta_j^{(r)})_{1,q}, \end{matrix}$$

$$\left.\begin{matrix} (u-\rho-\mu;\alpha_1,...,\alpha_r),(\alpha_j,\alpha_j^{'},...,\alpha_j^{(r)})_{1,p}:(a_j^{'},\alpha_j^{'})_{1,p^{'}}:...:(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}} \\ (u-v-\rho-\sigma-\mu-1;\alpha_1+\beta_1,...,\alpha_r+\beta_r),:(b_j^{'},\beta_j^{'})_{1,q^{'}}:...:(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}} \end{matrix}\right] \tag{4.1}$$

Provided all the conditions given with (3.1) and the conditions ([7],p.252-253, eq. (c.4), (c.5) and (c.6)) are satisfied.

For $n=0=p, q=0$, the multivariable H-function breaks up into a product of r H-function and consequently, (4.1) reduces to

$$(1-x)^{\rho-\frac{u}{2}}(1+x)^{\sigma+\frac{v}{2}}\prod_{i=1}^{r}\left\{H_{p_i,q_i}^{m_i,n_i}\left[(1-x)^{\alpha_i}(1+x)^{\beta_i}\left| \begin{matrix} a_j^{(i)},\alpha_j^{(i)} {}_{1,p_i} \\ b_j^{(i)},\beta_j^{(i)} {}_{1,q_i} \end{matrix}\right.\right]\right\}$$

$$=2^{\rho+\sigma}\sum_{N=0}^{\infty}\sum_{\mu=0}^{N}\frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_\mu}{N!\mu!\Gamma(1+v+N)\Gamma(1-u+\mu)}$$

$$P_{N-\frac{u-v}{2}}^{u,v}(x)H_{2,1:(p',q'):...:\,p^{(r)},q^{(r)}}^{0,2:(m',n'):...:\,m^{(r)},n^{(r)}}\left[\begin{matrix} 2^{\alpha_1+\beta_1} z_1 \\ \vdots \\ 2^{\alpha_r+\beta_r} z_r \end{matrix}\right| \begin{matrix} (-\sigma-v;\beta_1,...,\beta_r), \\ \\ - \end{matrix}$$

$$\left.\begin{matrix} (u-\rho-\mu;\alpha_1,...,\alpha_r):(a_j^{'},\alpha_j^{'})_{1,p^{'}}:...:(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}} \\ (u-v-\rho-\sigma-\mu-1;\alpha_1+\beta_1,...,\alpha_r+\beta_r),:(b_j^{'},\beta_j^{'})_{1,q^{'}}:...:(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}} \end{matrix}\right] \tag{4.2}$$

For $r=1$,(4.2) gives riseto the result due to Anandani [1].

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