

Quasi Swallowtail Singularities

Dedicated to the memory of Vladimir Zakalyukin

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Abstract: *We obtain a list of simple classes of singularities of function germs with respect to the quasi swallowtail equivalence relation. We discuss its connection with the singularities of Lagrangian projections in presence of a swallowtail. We also describe the bifurcation diagrams and caustics of simple quasi swallowtail singularities.*

Keywords: *Quasi border singularities, cusp, swallowtail, bifurcation diagram, caustic, Lagrangian projection.*

1 INTRODUCTION

In 2007, Vladimir Zakalyukin classified function germs with respect to new non-standard equivalence relations (see [1, 2]) which he named quasi equivalences. He was motivated by the needs of the theory of Lagrangian maps of singular varieties. In [2], the quasi relation is aimed to control positions of only critical points of functions defined on a space equipped with a smooth hypersurface and allow absolute freedom outside the critical locus. Later on and based on Zakalyukin's idea in [3, 4], more general settings were considered and similar equivalence relations were introduced for any hypersurface or a complete intersection which may be regular, singular or reducible. Such a hypersurface is called a border and the relation is called quasi border equivalence. In particular, for the hypersurfaces being a cylinder over a corner or a cusp, the lists of simple "in the sense of Arnold" classes of singularities of function germs with respect to quasi border equivalence relations were obtained there, respectively.

In spite of a rather artificial nature of the definitions, quasi border singularities have very natural applications. Their discriminants reflect the behavior of critical points of a function (for example, of its global extremum) on domains with borders.

Arnold's classical boundary function singularity with n parameters is related to a projection of a pair of n dimensional Lagrangian submanifolds which have $(n - 1)$ -dimensional regular intersection [5]. The quasi border equivalence relation for generating families of functions keeps information only on one Lagrangian submanifold of the pair and on its intersection with the second component. So, it is a more natural model for applications involving Lagrangian submanifolds with boundary. Allowing singular boundaries, we arrive at a notion of a *Lagrangian submanifold with a border*.

This is a pair (L, Γ) consisting of a Lagrangian submanifold $L^n \subset M = T^*\mathbb{R}^n$ and of an $(n - 1)$ -dimensional isotropic variety $\Gamma \subset L$. Such an object arises in various singularity theory applications to differential equations and variational problems. Isotropic submanifolds play the role of the initial data set with some inequality constraints.

The quasi border bifurcation diagrams of function germ deformations consist of two components, W_0 and W_1 . The W_0 is the ordinary discriminant which corresponds to all critical points of the deformation. The W_1 is a subset of W_0 which corresponds to the critical points on the border and satisfies extra equations defining the border. So the components have different dimensions. The caustic of a quasi border function deformation also consists of two strata or more. The first one is the ordinary caustic. The other stratum is the projection of the component W_1 of the bifurcation diagram to the base of the reduced deformation. The dimensions of these components of a caustic coincide. The geometry of bifurcation diagrams and caustics of simple quasi boundary and quasi corner singularities in \mathbb{R}^3 and \mathbb{R}^4 is described in [6].

The aim of the present paper is to consider a different type of a border. In particular, the border is cylinder over a swallowtail. The paper is organized as follows. In section 2 we recall the main definitions of the pseudo and quasi border equivalence relations from [4], and derive an expression of the quasi swallowtail tangent space. In section 3 we obtain the classifications of simple quasi swallowtail singularities. In section 4 the caustics and bifurcation diagrams of simple quasi swallowtail singularities are described. In section 5 we discuss the singularities of Lagrangian projections in presence of a swallowtail. Finally, in section 6 we finish the paper by a brief conclusion.

2 PSEUDO AND QUASI BORDER EQUIVALENCE RELATIONS

Consider a coordinate space $\mathbb{R}^n = \{w = (x, y)\}$ equipped with a cylindrical hypersurface Γ given by the equation $B(x) = 0$, where $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ and $y = (y_1, y_2, \dots, y_{n-m}) \in \mathbb{R}^{n-m}$. We shall call Γ a *border*.

The general statements below are valid for reasonably good borders. For rigorousness, we assume that the border is a stratified set, and the stratification satisfies the Whitney condition A.

We consider germs of C^∞ functions $f : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}$, in local coordinates w as above. We denote by \mathbf{C}_w the ring of all such germs at the origin and by $\mathcal{M}_{x,y}$ the maximal ideal in $\mathbf{C}_{x,y}$.

Definition 2.1 [4] Two functions $f_0, f_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ are called *pseudo border* equivalent if there exists a diffeomorphism $\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f_1 \circ \theta = f_0$, and if a critical point c of the function f_0 belongs to the border Γ then $\theta(c)$ also belongs to Γ and vice versa, if c is a critical point of f_1 and belongs to Γ then $\theta^{-1}(c)$ also belongs to Γ .

A similar definition can be introduced for germs of functions.

Remark 2.2 We assume in the definition 2.1 that if a critical point c belongs to some stratum of Γ then $\theta(c)$ belongs to the same stratum.

Clearly, the pseudo border equivalence is an equivalence relation: if $f_1 \sim f_2$ and $f_2 \sim f_3$ then $f_1 \sim f_3$. However, this relation is not a group action as the set of admissible diffeomorphisms depends on a function.

In the current paper, we consider the case when Γ is a cylinder over a swallowtail. The swallowtail is realized as the subset $\Gamma = \Gamma_{swl} \subset \mathbb{R}^3$ consisting of all points $\{(x_1, x_2, x_3)\}$ such that $z^4 + x_3z^2 + x_2z + x_1 = 0$ has multiple real roots. In this case the pseudo border equivalence will be also called pseudo swallowtail equivalence.

Recall that a vector field v preserves a border $\Gamma = \{B(x) = 0\}$ if the Lie derivative $L_v B$ belongs to the principal ideal generated by B . Such vector fields v are tangent to Γ . The module \mathbf{S}_Γ of all germs of C^∞ vector fields preserving a germ $(\Gamma, 0) \subset (\mathbb{R}^n, 0)$ is the Lie algebra of the group of diffeomorphisms of $(\mathbb{R}^n, 0)$ preserving $(\Gamma, 0)$. The module \mathbf{S}_Γ is called the *stationary algebra* of $(\Gamma, 0)$. Due to classical Zakalyukin’s algorithm [7] to write out basic vector fields tangent to the discriminant of an isolated function singularity, we have

$$\mathbf{S}_{\Gamma_{swl}} = \left\{ \left[4x_1h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 \right] \frac{\partial}{\partial x_1} + \left[3x_2h_1 + (4x_1 - x_3^2)h_2 - 2x_2x_3h_3 \right] \frac{\partial}{\partial x_2} + \left[2x_3h_1 + 3x_2h_2 + (4x_1 - x_3^2)h_3 \right] \frac{\partial}{\partial x_3} + \sum_{i=1}^{n-3} k_i \frac{\partial}{\partial y_i} : h_1, h_2, h_3, k_i \in \mathbf{C}_w \right\}.$$

Here $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y = (y_1, y_2, \dots, y_{n-3}) \in \mathbb{R}^{n-3}$.

Definition 2.3 [4] Let J be an ideal in \mathbf{C}_w , then we define the *radical* $Rad(J)$ of the ideal J as the set of all elements in \mathbf{C}_w , vanishing on the set of common zeros of germs from J :

$$Rad(J) = I(V(J)),$$

where

$$V(J) = \{w \in \mathbb{R}^n : h(w) = 0 \text{ for all } h \in J\},$$

and

$$I(V(J)) = \{\varphi \in \mathbf{C}_w : \varphi(w) = 0 \text{ for all } w \in V(J)\}.$$

Remark 2.4 In general, the radical of an ideal behaves badly when the ideal depends on a parameter [4].

Suppose that all function germs in a smooth family f_t are pseudo swallowtail equivalent to the function germ f_0 , $f_t \circ \theta_t = f_0$, $t \in [0, 1]$, with respect to a smooth family $\theta_t : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ of germs of diffeomorphisms such that $\theta_0 = id$ and $t \in [0, 1]$. Then we have the homological equation:

$$-\frac{\partial f_t}{\partial t} = \sum_{i=1}^3 \frac{\partial f_t}{\partial x_i} \dot{X}_i(t) + \sum_{j=1}^{n-3} \frac{\partial f_t}{\partial y_j} \dot{Y}_j(t),$$

where the vector field

$$v_t = \sum_{i=1}^3 \dot{X}_i(t) \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-3} \dot{Y}_j(t) \frac{\partial}{\partial y_j}$$

generates the phase flow θ_t .

Denote by $Rad\{I_t\}$ the radical of the gradient ideal I_t of the function f_t . Then by similar considerations to that in [2, 3, 4], the components of v_t satisfy the following:

$$\begin{aligned} \dot{X}_1(t) &\in \left\{ 4x_1h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 + Rad\{I_t\} \right\}, \\ \dot{X}_2(t) &\in \left\{ 3x_2h_1 + (4x_1 - x_3^2)h_2 - 2x_2x_3h_3 + Rad\{I_t\} \right\}, \\ \dot{X}_3(t) &\in \left\{ 2x_3h_1 + 3x_2h_2 + (4x_1 - x_3^2)h_3 + Rad\{I_t\} \right\}, \end{aligned}$$

and $\dot{Y}_i(t) \in \mathbf{C}_w$, where $h_1, h_2, h_3 \in \mathbf{C}_w$.

Due to Remark 2.4 we modify the pseudo equivalence relation to have a better parameter dependence. Namely, we replace $Rad\{I_t\}$ by the ideal I_t itself in the equivalence definition. This provides us the notion of *the quasi swallowtail equivalence*.

Definition 2.5 Two functions $f_0, f_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ are called *quasi swallowtail equivalent*, if they are pseudo swallowtail equivalent and there is a family of functions f_t which depends continuously on parameter $t \in [0, 1]$ and a continuous piece-wise smooth family of diffeomorphisms $\theta_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ also depending on $t \in [0, 1]$ such that: $f_t \circ \theta_t = f_0$, $\theta_0 = id$ and the components of the vector field v_t generating θ_t on each segment of smoothness satisfy the following:

$$\begin{aligned} \dot{X}_1(t) &\in \left\{ 4x_1h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 + \{I_t\} \right\}, \\ \dot{X}_2(t) &\in \left\{ 3x_2h_1 + (4x_1 - x_3^2)h_2 - 2x_2x_3h_3 + \{I_t\} \right\}, \\ \dot{X}_3(t) &\in \left\{ 2x_3h_1 + 3x_2h_2 + (4x_1 - x_3^2)h_3 + \{I_t\} \right\}, \end{aligned}$$

and $\dot{Y}_i(t) \in \mathbf{C}_w$, where $h_1, h_2, h_3 \in \mathbf{C}_w$.

Remarks 2.6

1. Such a family θ_t of diffeomorphisms generated by the vector field v_t as well as the vector field itself will be called *admissible* for the family f_t .
2. The tangent space $TQSWL_f$ to the quasi swallowtail equivalence class of f has the following description:

$$\begin{aligned} TQSWL_f &= \left\{ \frac{\partial f}{\partial x_1} \left(4x_1h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 + \sum_{i=1}^3 \frac{\partial f}{\partial x_i} A_i \right) \right. \\ &+ \frac{\partial f}{\partial x_2} \left(3x_2h_1 + (4x_1 - x_3^2)h_2 - 2x_2x_3h_3 + \sum_{i=1}^3 \frac{\partial f}{\partial x_i} B_i \right) \\ &+ \frac{\partial f}{\partial x_3} \left(2x_3h_1 + 3x_2h_2 + (4x_1 - x_3^2)h_3 + \sum_{i=1}^3 \frac{\partial f}{\partial x_i} C_i \right) \\ &\left. + \sum_{i=1}^{n-3} \frac{\partial f}{\partial y_i} D_i \quad : \quad h_1, h_2, h_3, A_i, B_i, C_i, D_i \in \mathbf{C}_w \right\}. \end{aligned}$$

Due to the inclusion $I_0^2 \subset TQSWL_f \subset I_0$, where I_0 is the gradient ideal of f , we have

Proposition 2.7 *A function germ f has a finite codimension with respect to the quasi swallowtail equivalence if and only if f has a finite codimension with respect to the right equivalence.*

Definition 2.8 Two function germs are said to be *stably quasi swallowtail equivalent* if they become quasi swallowtail equivalent after the addition of non-degenerate quadratic forms in an appropriate number of extra cylindrical variables.

2.1 Basic Techniques of the Classification and Prenormal Forms

We will use Moser’s homotopy method and the following technique which is essential for our normal form reduction in Section 3 to establish quasi swallowtail equivalence between function germs. It is similar to Lemma 8.1 in [3] and Lemma 2.10 in [4].

Let us fix a convenient Newton diagram $\Delta \subset \mathbb{Z}_{\geq 0}^n$. The ideals S_γ of function germs of the Newton order at least γ , $\gamma \geq 0$, equip the ring \mathbf{C}_w with the Newton filtration: $S_0 = \mathbf{C}_w$, $S_\delta \supset S_\gamma$ if $\delta < \gamma$ [8]. We assume here that the scaling factor for the orders is chosen so that functions with the Newton diagram Δ have order N .

Let $f = f_0 + f_*$ be a decomposition of a function germ f into its principal part f_0 of the Newton degree N and higher order terms f_* . We assume that f_0 has a finite codimension with respect to the right equivalence.

Lemma 2.9 *Consider a monomial basis of the linear space $\mathbf{C}_w/TQSWL_{f_0}$. Let $e_1(w), e_2(w), \dots, e_s(w)$ be all its elements of Newton degrees higher than N .*

Suppose that for any $\varphi \in S_\gamma \setminus S_{>\gamma}$, $\gamma > N$:

1. There is an admissible vector field $\dot{w} = \sum \dot{w}_i \frac{\partial}{\partial w_i}$ where $\dot{w} = (\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{y}_1, \dot{y}_2, \dots, \dot{y}_{n-3})$,

$$\dot{x}_1 = 4x_1h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 + \sum_{i=1}^n A_{1,i} \frac{\partial f_0}{\partial w_i}, \quad \dot{x}_2 = 3x_2h_1 + (4x_1 - x_3^2)h_2 - 2x_2x_3h_3 + \sum_{i=1}^n A_{2,i} \frac{\partial f_0}{\partial w_i},$$

$$\dot{x}_3 = 2x_3h_1 + 3x_2h_2 + (4x_1 - x_3^2)h_3 + \sum_{i=1}^n A_{3,i} \frac{\partial f_0}{\partial w_i}, \quad \text{and} \quad \dot{y}_1, \dots, \dot{y}_{n-3} \in \mathbf{C}_w,$$

with $h_1, h_2, h_3, A_{1,i}, A_{2,i}, A_{3,i} \in \mathbf{C}_w$, such that

$$\varphi = \sum_{i=1}^n \frac{\partial f_0}{\partial w_i} \dot{w}_i + \hat{\varphi} + \sum_{i=1}^s c_i e_i(w),$$

where $\hat{\varphi} \in S_{>\gamma}$ and $c_i \in \mathbb{R}$.

2. Moreover, for any δ , $N < \delta < \gamma$, and any $\psi \in S_\delta$ the expression

$$E(\psi, \varphi) = \sum_{i=1}^3 \frac{\partial \psi}{\partial x_i} \left[\dot{x}_i + \sum_{j=1}^n A_{i,j} \frac{\partial \psi}{\partial w_j} \right] + 2 \sum_{i=1}^3 \frac{\partial f_0}{\partial x_i} \left[\sum_{j=1}^n A_{i,j} \frac{\partial \psi}{\partial w_j} \right] + \sum_{i=1}^{n-3} \frac{\partial \psi}{\partial y_i} \dot{y}_i$$

belongs to S_γ .

Then any germ $f = f_0 + f_*$ is quasi swallowtail equivalent to a germ $f_0 + \sum_{i=1}^s a_i e_i$, where $a_i \in \mathbb{R}$.

Remark 2.10 A version of the Lemma is also valid for functions with the Newton principal part f_0 of infinite right equivalence codimension, which is the same as having $\dim(\mathbf{C}_w/TQSWL_{f_0}) = \infty$. Namely, still assuming the Newton degree of f_0 being N , let $e_1(w), \dots, e_s(w)$ be the degrees higher than N part of a monomial basis of $\mathbf{C}_w/(TQSWL_{f_0} + S_M)$ for some $M > N$. Assume the conditions of Lemma 2.9 hold for all $\gamma < M$. Then, any function with the Newton principal part f_0 is quasi swallowtail equivalent to

$$f_0 + \sum_{i=1}^s a_i e_i + \Psi, \quad \text{where } \Psi \in S_M.$$

2.2 Quadratic Terms

Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ of the form

$$f(x, y) = f_2(x, y) + f_3(x, y),$$

where f_2 is a quadratic form in x and y , and $f_3 \in \mathcal{M}_{x,y}^3$.

Remark 2.11 A version of Lemma 2.12, Lemma 2.13, Lemma 2.14 and Lemma 2.15 in [4] are also true for functions when the cuspidal edge replaced by Γ_{swl} . In fact, the components of the vector fields used there in their proofs are only from the gradient ideal of a family of functions. The phase flow of such vector fields leaves all critical points of the family fixed. Therefore, the family of diffeomorphisms generated by such vector fields is also admissible for quasi swallowtail equivalence. We state them below for Γ_{swl} with details of their proofs.

Lemma 2.12 Let $f : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0)$ be a function germ at the origin in local coordinates x_1, x_2 and x_3 only. If f_2 is a non-degenerate quadratic form then f is quasi swallowtail equivalent to $\pm x_1^2 \pm x_2^2 \pm x_3^2$.

Proof. Since f_2 is a non-degenerate quadratic form, vector fields with components from the gradient ideal of a function with a non-degenerate quadratic part are all vector fields vanishing at the origin. Therefore any family of diffeomorphisms preserving the origin is admissible, and the Lemma follows from the standard Morse Lemma. \square

Let $n \geq 3$ and set $f^*(y) = f|_{x=0}$. Denote by r^* the rank of the second differential $d_0^2 f^*$ at the origin. Set $c = n - 3 - r^*$. Denote by r the rank of the second differential $d_0^2 f$ at the origin.

Lemma 2.13 (Stabilization) The function germ $f(x, y)$ is quasi swallowtail equivalent to a germ $\sum_{i=1}^{r^*} \pm y_i^2 + g(x, \tilde{y})$, where $\tilde{y} \in \mathbb{R}^c$ and $g^* \in \mathcal{M}_{\tilde{y}}^3$. For quasi swallowtail equivalent germs f , the respective reduced germs g are quasi swallowtail equivalent.

Proof. Up to a linear transformation in y , we have

$$f = \sum_{i=1}^{r^*} \pm y_i^2 + \sum_{i=1}^{n-3} \sum_{j=1}^3 a_{i,j} y_i x_j + Q_2(x) + f_3(x, y), \tag{1}$$

with $f_3 \in \mathcal{M}_{x,y}^3$ and Q_2 a quadratic form in x only.

Let $\hat{y} = (y_1, y_2, \dots, y_{r^*})$ and $\tilde{y} = (y_{r^*+1}, \dots, y_{n-3}) \in \mathbb{R}^{n-3-r^*}$. Then, (1) can be written as

$$f_1 = \sum_{i=1}^{r^*} \pm y_i^2 + \varphi(x, \hat{y}, \tilde{y}) + \tilde{f}(x, \tilde{y}),$$

where

$$\varphi = \sum_{i=1}^{r^*} \sum_{j=1}^3 a_{i,j} y_i x_j + \sum_{l=1}^{r^*} y_l \tilde{\varphi}_l(x, y) \quad \text{with} \quad \tilde{\varphi}_l \in \mathcal{M}_{x,y}^2,$$

and

$$\tilde{f} = Q_2(x) + \sum_{i=r^*+1}^{n-3} \sum_{j=1}^3 a_{i,j} y_i x_j + \tilde{f}_3(x, \tilde{y}) \quad \text{with} \quad \tilde{f}_3 \in \mathcal{M}_{x,\tilde{y}}^3.$$

We now aim to find a family

$$\theta_t : (x, y) \mapsto (x, \hat{Y}_t(x, \hat{y}), \tilde{y})$$

of diffeomorphisms which eliminates φ .

Take a family $f_t = \sum_{i=1}^{r^*} \pm y_i^2 + t\varphi(x, \hat{y}, \tilde{y}) + \tilde{f}_t(x, \tilde{y})$ which joins f_1 and $f_0 = \sum_{i=1}^{r^*} \pm y_i^2 + \tilde{f}_0(x, \tilde{y})$ with $t \in [0, 1]$ and $\tilde{f} = \tilde{f}_1$. Here, \tilde{f}_t and \tilde{f}_0 are unknown. So, we want to solve the homological equation for \dot{y} and simultaneously for \dot{f}_t .

The homological equation takes the form

$$-\frac{\partial f_t}{\partial t} = \sum_{i=1}^3 \frac{\partial f_t}{\partial x_i} \dot{x}_i + \sum_{i=1}^{r^*} \frac{\partial f_t}{\partial y_i} \dot{y}_i + \sum_{j=r^*+1}^{n-3} \frac{\partial f_t}{\partial y_j} \dot{y}_j. \tag{2}$$

Note that $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = \dot{y}_j = 0, j = r^* + 1, \dots, n - 3$, as x and \tilde{y} do not change with t .

Thus, (2) can be written as

$$-(\varphi + \frac{\partial \tilde{f}_t}{\partial t}) = \sum_{i=1}^{r^*} (\pm 2y_i + t \frac{\partial \varphi}{\partial y_i}) \dot{y}_i. \tag{3}$$

Set $z_i = \pm 2y_i + t \frac{\partial \varphi}{\partial y_i}, i = 1, \dots, r^*$, which are known functions. Note that the matrix $(\frac{\partial z_i}{\partial \tilde{y}_j})$ has the maximal rank at the origin for any value of t . Hence we can take $z = (z_1, z_2, \dots, z_{r^*})$ as new coordinates instead of \hat{y} . Thus, (3) takes the form

$$-(\varphi + \frac{\partial \tilde{f}_t}{\partial t}) = \sum_{i=1}^{r^*} z_i \dot{y}_i.$$

Using the Hadamard Lemma, we write this as

$$\sum_{i=1}^{r^*} z_i \psi_i(x, z, \tilde{y}, t) + \phi(x, \tilde{y}, t) + \frac{\partial \tilde{f}_t}{\partial t} = \sum_{i=1}^{r^*} -z_i \dot{y}_i.$$

By taking $\psi_i = -\dot{y}_i$ and $\frac{\partial \tilde{f}_t}{\partial t} = -\phi$, we show that the homological equation is solvable.

The last step is to find \tilde{f}_0 . This can be done using the relation

$$-\int_0^1 \phi dt = \int_0^1 \frac{\partial \tilde{f}_t}{\partial t} dt = \tilde{f}_1 - \tilde{f}_0.$$

Note that the vector field $\dot{v} = \sum_{i=1}^{r^*} \dot{y}_i \frac{\partial}{\partial y_i}$ is defined in some neighborhood of the segment $[0, 1]$ of the t -axis in the space $\mathbb{R}^n \times \mathbb{R}_t$, which is due to the z_i vanishing on this segment.

Hence all the f_t are quasi swallowtail equivalent. In particular, the function germ f_1 is quasi swallowtail equivalent to f_0 .

The second claim of the Lemma can be deduced directly as the family

$$\theta_t : (x, y) \mapsto (x, \hat{Y}_t(x, \hat{y}), \tilde{y})$$

preserves the projection $(x, \hat{y}, \tilde{y}) \mapsto (x, \tilde{y})$. □

Lemma 2.14 *Let $f : (\mathbb{R}^n, 0) \rightarrow \mathbb{R}$ be a function germ with an isolated critical point at the origin, and I_0 its gradient ideal. Then f is quasi swallowtail equivalent for each $t \in [0, 1]$ to the function germ $g_t(w) = f(w) + th(w)$ with $h(w) \in I_0^2$, provided that the rank r of the second differential $d_0^2 g_t$ of g_t at the origin is constant.*

Proof. At first we claim that if the rank of $d_0^2 g_t$ is constant then for different t the gradient ideals I_t of g_t coincide. Since the claim does not depend on the choice of local coordinates, we may assume that the quadratic part of f at the origin has diagonal form $\sum_{i=1}^r \varepsilon_i w_i^2$, where $\varepsilon_i = \pm 1$ for $i = 1, \dots, r$. We also set $\varepsilon_i = 0$ for $i > r$.

The quadratic part of g_t at the origin is

$$\sum_{i,j=1}^r (\varepsilon_i \delta_{ij} w_i^2 + 4th_{ij}(0) \varepsilon_i \varepsilon_j w_i w_j) = \sum_{i,j=1}^r D_{ij} w_i w_j$$

where the h_{ij} , $i, j = 1, \dots, n$ are the coefficients of the decomposition

$$h(w) = \sum_{i,j=1}^n h_{ij}(w) \frac{\partial f}{\partial w_i} \frac{\partial f}{\partial w_j}$$

of the function h , δ_{ij} is the Kronecker symbol, and $D_{ij} = \varepsilon_i \delta_{ij} + 4t \varepsilon_i \varepsilon_j h_{ij}(0)$. We will assume here that $h_{ij} = h_{ji}$.

The $r \times r$ matrix with entries D_{ij} is invertible for any t since the rank of $d_0^2 g_t$ is r . Reversing signs of some of its rows, we see that the $n \times n$ matrix with the entries $\hat{D}_{ij} = \delta_{ij} + 4t\varepsilon_j h_{ij}(0)$, for $i, j = 1, \dots, r$ and $\hat{D}_{ij} = \delta_{ij}$ otherwise, is also invertible.

The differentiation

$$\frac{\partial g_t}{\partial w_i} = \frac{\partial f}{\partial w_i} + t \sum_{k,j=1}^n \left(2h_{kj} \frac{\partial^2 f}{\partial w_k \partial w_i} + \frac{\partial h_{kj}}{\partial w_i} \frac{\partial f}{\partial w_k} \right) \frac{\partial f}{\partial w_j}$$

implies that $I_t \subset I_0$. This derivative can also be written as

$$\frac{\partial g_t}{\partial w_i} = \sum_{j=1}^n (\delta_{ij} + 4t\varepsilon_i h_{ij}(0) + R_{ji}) \frac{\partial f}{\partial w_j} = \sum_{j=1}^n (\hat{D}_{ji} + R_{ji}) \frac{\partial f}{\partial w_j},$$

where the functions R_{ij} vanish at $w = 0$. So in some neighborhood of the interval $[0, 1]$ of the t -axis the matrix $(\hat{D}_{ji} + R_{ji})$ is invertible. This implies that $I_0 \subset I_t$. Hence, $I_t = I_0$.

Now the homological equation $-\frac{\partial g_t}{\partial t} = \sum_{i=1}^n \frac{\partial g_t}{\partial w_i} V_i$ can be solved for the unknown functions V_i which belong to the gradient ideal I_t for any t , since the left hand side belongs to the square of this ideal. The phase flow of the vector field $\sum V_i \frac{\partial}{\partial w_i}$ leaves all critical points of g_t fixed. Hence all the germs g_t are quasi swallowtail equivalent. \square

Lemmas 2.13 and 2.14 imply the following improved stabilization splitting.

Lemma 2.15 *There is a non-negative integer $s \leq r - r^*$ such that the function germ $f(x, y)$ is quasi swallowtail equivalent to $\sum_{i=1}^{r^*+s} \pm y_i^2 + \tilde{f}(x, \tilde{y})$, where $\tilde{y} \in \mathbb{R}^{c-s}$ and \tilde{f} is a sum of a function germ from $\mathcal{M}_{x, \tilde{y}}^3$ and a quadratic form in x only. For quasi swallowtail equivalent germs f , the corresponding reduced germs \tilde{f} are quasi swallowtail equivalent.*

Proof. Due to Lemma 2.13, we can assume that the quadratic part of the function is

$f_2 = \sum_{i=1}^{r^*} \pm y_i^2 + x_1 \sum_{i=r^*+1}^{n-3} \alpha_{1,i} y_i + x_2 \sum_{i=r^*+1}^{n-3} \alpha_{2,i} y_i + x_3 \sum_{i=r^*+1}^{n-3} \alpha_{3,i} y_i + g_2(x)$ with constant coefficients $\alpha_{j,i}$ and the quadratic form g_2 in x only. Suppose that some of these coefficients, for example α_{1,r^*+1} , is non-zero. Then, summing up the function f with $\delta \left(\frac{\partial f}{\partial x_1} \right)^2$ for sufficiently small δ gives a new function g which (according to Lemma 2.14) is quasi swallowtail equivalent to f and contains the term $y_{r^*+1}^2$ with a non-zero coefficient. Therefore the rank of the restriction of g to the $x = 0$ subspace is larger than r^* . Repeating the procedure several times, if needed, we get a function germ with a larger value of r^* and without the $x_j y_{>r^*}$ terms. This is exactly the form required. \square

3 CLASSIFICATION OF SIMPLE FUNCTIONS

Following [8], we call a function germ *simple* if its sufficiently small neighbourhood in the space of all function germs contains only a finite number of quasi equivalence classes.

The quasi swallowtail classification of critical points outside the border Γ_{swl} coincides with the standard right equivalence. Hence the standard classes A_k, D_k, E_6, E_7 and E_8 form the list of simple classes in this case. Also by definition, non-critical points are all equivalent wherever they are. When the critical point of a function germ lies on Γ_{swl} then we distinguish the following cases:

- If the base point is at a regular point of Γ_{swl} , then the quasi swallowtail equivalence coincides with the quasi boundary equivalence. Hence, the list of simple quasi swallowtail classes in this case is the same as that of simple quasi boundary classes [2].
- If the base point belongs to the self-interesting stratum of Γ_{swl} , then the quasi swallowtail equivalence coincides with the quasi corner equivalence. Hence, the list of simple quasi swallowtail classes in this case is the same as that of simple quasi corner classes[3].
- If the base point belongs to the cusp stratum of Γ_{swl} , then the list of simple quasi swallowtail classes is the same as that of simple quasi corner classes [4].
- The remaining case of a function germ having a swallowtail point is described by the following theorem.

Theorem 3.1 *Let a function germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$, be simple with respect to the quasi swallowtail equivalence. Then, either its quadratic part f_2 is non-degenerate and hence f is quasi swallowtail equivalent to $\mathcal{Q}_2 : \pm x_1^2 \pm x_2^2 \pm x_3^2 + \sum_{i=1}^{n-3} \pm y_i^2$ or f_2 has corank 1 in which case f is stably quasi swallowtail equivalent to one of the following simple classes:*

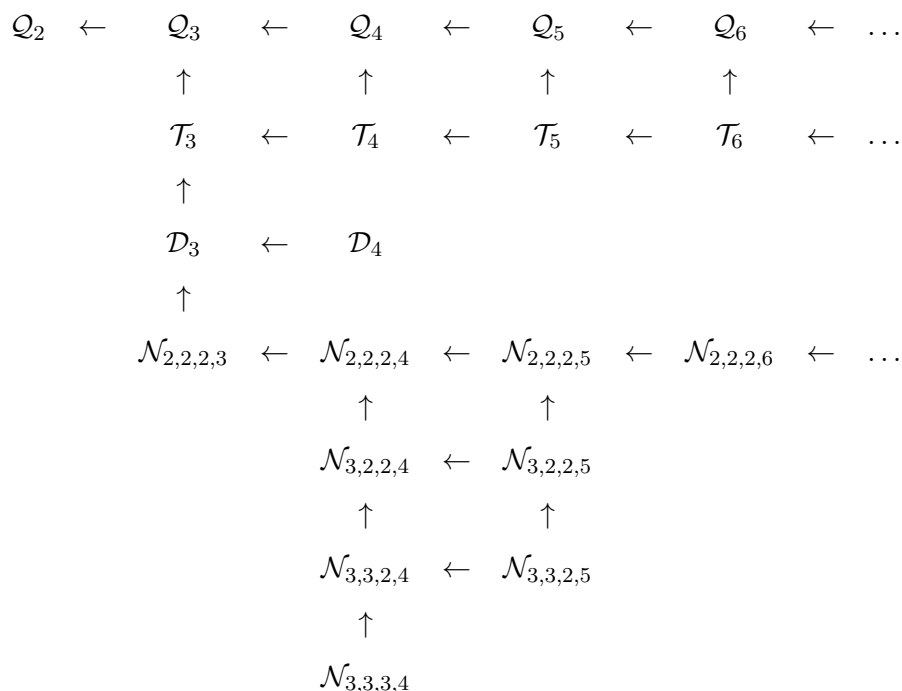
Notation	Normal form	Restrictions	Codimension
\mathcal{Q}_k	$\pm x_2^2 \pm x_3^2 \pm x_1^k$	$k \geq 3$	$k + 2$
\mathcal{T}_k	$\pm x_1^2 \pm x_3^2 \pm x_2^k$	$k \geq 3$	$k + 3$
\mathcal{D}_3	$\pm x_1^2 \pm x_2^2 + x_3^3$		7
$\mathcal{N}_{2,2,2,k}$	$\pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm (x_3 + y_1^2)^2 \pm y_1^k$	$k \geq 3$	$k + 5$
$\mathcal{N}_{3,2,2,4}$	$\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^2)^2 \pm (x_3 + y_1^2)^2 \pm y_1^4$		10
$\mathcal{N}_{3,3,2,4}$	$\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_3 + y_1^2)^2 \pm y_1^4$		11
$\mathcal{N}_{3,3,3,4}$	$\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_3 + y_1^3)^2 \pm y_1^4$		12

Remarks 3.2

1. Any germ f with the quadratic part of corank greater than 1 is non-simple.
2. Any germ of corank 1 is either simple (and hence quasi swallowtail equivalent to one of the germs in the above theorem) or belongs to a subset of infinite codimension in the space of all germs.
3. The fencing classes are stabilizations of the following:

Notation	Class	Restrictions	Codimension
$\tilde{\mathcal{Q}}$	$\pm(x_1 + x_2 + x_3)^2 \pm x_1^3 + \alpha x_3^3$	$\alpha \in \mathbb{R} \setminus \{0\}$	8
\mathcal{D}_4	$\pm x_2^2 \pm (x_1 + \delta x_3^2)^2 \pm x_3^4,$	$\delta \in \mathbb{R}$	8
$\mathcal{N}_{3,2,2,5}$	$\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^2)^2 + \pm(x_2 + y_1^2)^2 + \beta y_1^5$	$\beta \in \mathbb{R} \setminus \{0\}$	11
$\mathcal{N}_{3,3,2,5}$	$\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_2 + y_1^2)^2 \gamma y_1^5$	$\gamma \in \mathbb{R} \setminus \{0\}$	12

4. The graph of adjacencies in low codimension is as follows:



Also, $\mathcal{D}_3 \leftarrow \tilde{\mathcal{Q}}$.

To prove Theorem 3.1, we need the following auxiliary results.

Lemma 3.3 *Let $\kappa = n - r$ be the corank of the second differential $d_0^2 f$ at the origin.*

1. If $\kappa = 0$, then f is quasi swallowtail equivalent to $\sum_{i=1}^{n-3} \pm y_i^2 + f_2(x) + f_3(x)$, where f_2 is a non-degenerate quadratic form and $f_3 \in \mathcal{M}_x^3$.
2. If $\kappa = 1$, then f is quasi swallowtail equivalent to either $\sum_{i=1}^{n-3} \pm y_i^2 + \tilde{f}(x)$ with $\text{rank}(d_0^2 \tilde{f}) = 2$ or to $\sum_{i=2}^{n-3} \pm y_i^2 \pm x_1^2 \pm x_2^2 \pm x_3^2 + f_3(x, y_1)$, where $f_3(x, y_1) \in \mathcal{M}_{x, y_1}^3$.
3. If $\kappa \geq 2$, then f is non-simple.

Proof. Lemmas 2.12 and 2.15 provide the first two parts of Lemma 3.3.

For part 3, suppose that $\kappa = 2$. Then Lemma 2.15 yields that any function germ $f(x, y)$ reduced to one of the following forms:

0. $F_0 = \sum_{i=1}^{n-3} \pm y_i^2 + f_2(x) + f_3(x)$ where $f_3 \in \mathcal{M}_x^3$, and f_2 is a quadratic form in x of rank one, or
1. $F_1 = \sum_{i=2}^{n-3} \pm y_i^2 + f_2(x) + f_3(x, \tilde{y}_1)$ where $f_3 \in \mathcal{M}_{x, \tilde{y}_1}^3$, $\tilde{y}_1 \in \mathbb{R}$ and f_2 is a quadratic form in x of rank two, or
2. $F_2 = \sum_{i=3}^{n-3} \pm y_i^2 + f_2(x) + f_3(x, \tilde{y})$ where $f_3 \in \mathcal{M}_{x, \tilde{y}}^3$, $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ and f_2 is a non-degenerate

quadratic form in x .

Consider the germ F_0 . Then we may write $F_0 = \sum_{i=1}^{n-3} \pm y_i^2 \pm (ax_1 + bx_2 + cx_3)^2 + f_3$ where $f_3 \in \mathcal{M}_x^3$, and $a, b, c \in \mathbb{R}$ (a, b , and c are not all zeros simultaneously). Suppose $a \neq 0$ and set $\tilde{x}_1 = ax_1 + bx_2 + cx_3$. Hence we can take \tilde{x}_1 as a new coordinate instead of x_1 . Thus, up to linear transformation in \tilde{x}_1, x_2 and x_3 , we have

$$F_0 \sim \tilde{F} = \sum_{i=1}^{n-3} \pm y_i^2 \pm \tilde{x}_1^2 + \tilde{f}_3, \quad \tilde{f}_3 \in M_{\tilde{x}_1, x_2, x_3}^3.$$

The tangent space to the quasi swallowtail orbit at the germ \tilde{F} in the new coordinates is

$$\begin{aligned} TQSWL_{\tilde{F}} = & \left\{ \frac{\partial \tilde{F}}{\partial \tilde{x}_1} \left[\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3)h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 + \frac{\partial \tilde{F}}{\partial \tilde{x}_1}A_1 + \sum_{i=2}^3 \frac{\partial \tilde{F}}{\partial x_i}A_i \right] \right. \\ & + \frac{\partial \tilde{F}}{\partial x_2} \left[3x_2h_1 + \left(\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3) - x_3^2 \right)h_2 - 2x_2x_3h_3 + \frac{\partial \tilde{F}}{\partial \tilde{x}_1}B_1 + \sum_{i=2}^3 \frac{\partial \tilde{F}}{\partial x_i}B_i \right] \\ & \left. + \frac{\partial \tilde{F}}{\partial x_3} \left[2x_3h_1 + 3x_2h_2 + \left(\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3) - x_3^2 \right)h_3 + \frac{\partial \tilde{F}}{\partial \tilde{x}_1}C_1 + \sum_{i=2}^3 \frac{\partial \tilde{F}}{\partial x_i}C_i \right] \right\}. \end{aligned}$$

Now, consider the principal part $\tilde{F}_0 = \pm \tilde{x}_1$ of \tilde{F} . The tangent space to the quasi swallowtail orbit at \tilde{F}_0 is

$$TQSWL_{\tilde{F}_0} = \left\{ \pm 2\tilde{x}_1 \left[\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3)h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 + \frac{\partial \tilde{F}}{\partial \tilde{x}_1}A_1 + \sum_{i=2}^3 \frac{\partial \tilde{F}}{\partial x_i}A_i \right] \right\}.$$

Thus, we get $mod TQSWL_{\tilde{F}_0} : \tilde{x}_1^2 \equiv 0, \tilde{x}_1(-bx_2 - cx_3) \equiv 0, \tilde{x}_1x_2x_3 \equiv 0$, and $\tilde{x}_1x_2^2 \equiv 0$. Therefore, we have $\mathbf{C}_{\tilde{x}_1, x_2, x_3} / TQSWL_{\tilde{F}_0} \simeq \left\{ \alpha \tilde{x}_1 + \beta \tilde{x}_1x_2 + \varphi(x_2, x_3) : \alpha, \beta \in \mathbb{R}, \varphi \in \mathbf{C}_{x_2, x_3} \right\}$. According to Remark 2.10, the germ \tilde{F} is quasi swallowtail equivalent to

$$\pm \tilde{x}_1^2 + g(x_2, x_3) \quad \text{with } g \in \mathcal{M}_{x_2, x_3}^3. \quad (*)$$

Let $H = \tilde{a}\tilde{x}_1^2 + a_{30}x_2^3 + a_{21}x_2^2x_3 + a_{12}x_2x_3^2 + a_{03}x_3^3$ be the lowest quasi homogenous part of $(*)$ with respect to the weights: $w_{\tilde{x}_1} = 3, w_{x_2} = w_{x_3} = 2$. The tangent space to the quasi swallowtail orbit at H is

$$\begin{aligned} TQSWL_H = & \left\{ \frac{\partial H}{\partial \tilde{x}_1} \left[\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3)h_1 - \frac{x_2x_3}{2}h_2 - \frac{3x_2^2}{4}h_3 + \frac{\partial H}{\partial \tilde{x}_1}A_1 + \sum_{i=2}^3 \frac{\partial H}{\partial x_i}A_i \right] \right. \\ & + \frac{\partial H}{\partial x_2} \left[3x_2h_1 + \left(\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3) - x_3^2 \right)h_2 - 2x_2x_3h_3 + \frac{\partial H}{\partial \tilde{x}_1}B_1 + \sum_{i=2}^3 \frac{\partial \tilde{F}}{\partial x_i}B_i \right] \\ & \left. + \frac{\partial H}{\partial x_3} \left[2x_3h_1 + 3x_2h_2 + \left(\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3) - x_3^2 \right)h_3 + \frac{\partial H}{\partial \tilde{x}_1}C_1 + \sum_{i=2}^3 \frac{\partial H}{\partial x_i}C_i \right] \right\}. \end{aligned}$$

The terms of H in $TQSWL_{f_2}$ are from

$$\begin{aligned} & \left[\left(\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3) \right) \frac{\partial H}{\partial \tilde{x}_1} + 3x_2 \frac{\partial H}{\partial x_2} + 2x_3 \frac{\partial H}{\partial x_3} \right] h_1, \\ & \left[-\frac{x_2x_3}{2} \frac{\partial H}{\partial \tilde{x}_1} + \left(\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3) - x_3^2 \right) \frac{\partial H}{\partial x_2} + 3x_2 \frac{\partial H}{\partial x_3} \right] h_2, \\ & \left[-\frac{3x_2^2}{4} \frac{\partial H}{\partial \tilde{x}_1} - 2x_2x_3 \frac{\partial H}{\partial x_2} + \left(\frac{4}{a}(\tilde{x}_1 - bx_2 - cx_3) - x_3^2 \right) \frac{\partial H}{\partial x_3} \right] h_3, \end{aligned}$$

and

$$\left(\frac{\partial H}{\partial \tilde{x}_1} \right)^2 A_1.$$

These terms form a subspace of dimension at most 4. The dimension of the space $L = \{(\tilde{a}, a_{30}, a_{21}, a_{12}, a_{03})\}$ of coefficients of H is 5 which is greater than the subspace dimension. This means that the germ F_0 is non-simple.

By similar arguments we can show that F_0 is non-simple when $a = 0$ and $b \neq 0$ or $a = b = 0$ and $c \neq 0$.

In the next case we have $F_1 = \sum_{i=2}^{n-3} \pm y_i^2 \pm (a_1x_1 + b_1x_2 + c_1x_3)^2 + (a_2x_1 + b_2x_2 + c_2x_3)^2 + f_3$, where $f_3 \in \mathcal{M}_{x,\tilde{y}_1}$, and $a_i, b_i, c_i \in \mathbb{R}$ (a_1, b_1, c_1 or a_2, b_2, c_2 are not all zeros simultaneously). Note that F_1 deforms to

$$\tilde{F}_1 = \sum_{i=2}^{n-3} \pm y_i^2 \pm (a_1x_1 + b_1x_2 + c_1x_3)^2 + (a_2x_1 + b_2x_2 + c_2x_3 + \delta\tilde{y}_1)^2 + f_3, \quad \delta \neq 0.$$

According to Lemma 2.15, \tilde{F}_1 is quasi swallowtail equivalent to $\sum_{i=1}^{n-3} \pm y_i^2 + \pm(ax_1 + bx_2 + cx_3)^2 + \tilde{f}$ with $\tilde{f} \in \mathcal{M}_x^3$, which we have already shown to be non-simple. Similar argument yields that F_2 is adjacent to F_1 and the result follows. \square

Lemma 3.4 *Let $f : (\mathbb{R}^3, 0) \mapsto (\mathbb{R}, 0)$ be a function germ in local coordinates x_1, x_2 and x_3 , and with a critical point at the origin. If the quadratic form f_2 of f has rank 2 then f is quasi swallowtail equivalent to either $\pm x_2^2 \pm x_3^2 + \varphi_1(x_1)$ where $\varphi_1 \in \mathcal{M}_{x_1}^3$ or $\pm x_1^2 \pm x_3^2 + \varphi_2(x_2)$ where $\varphi_2 \in \mathcal{M}_{x_2}^3$ or $\pm x_1^2 \pm x_2^2 + x_1\varphi_3(x_3) + \varphi_4(x_3)$ where $\varphi_3 \in \mathcal{M}_{x_3}^2$ and $\varphi_4 \in \mathcal{M}_{x_3}^3$.*

Proof. The hierarchy of lines in $T_0\mathbb{R}^3$ with respect to diffeomorphisms preserving the swallowtail is

$$x_1\text{-axis} \leftarrow x_2\text{-axis} \leftarrow x_3\text{-axis}.$$

Considering them as kernals of quadratic forms, we get the hierarchy of corank 2 forms

$$\pm x_2^2 \pm x_3^2 \leftarrow \pm x_1^2 \pm x_3^2 \leftarrow \pm x_1^2 \pm x_2^2.$$

Consider the germ $G = \pm x_2^2 \pm x_3^2 + g_3(x)$, where $g_3 \in \mathcal{M}_x^3$. Let $G_0 = \pm x_2^2 \pm x_3^2$. The quasi swallowtail tangent space at G_0 is

$$\begin{aligned} TQSWL_{G_0} = & \left\{ \pm 2x_2 [3x_2h_1 + (4x_1 - x_3^2)h_2 - 2x_2x_3h_3 + x_2B_2 + x_3B_3] \right. \\ & \left. \pm 2x_3 [2x_3h_1 + 3x_2h_2 + (4x_1 - x_3^2)h_3 + x_2C_2 + x_3C_3] \right\}. \end{aligned}$$

Thus, we get $\text{mod } TQSWL_{G_0} : x_2^2 \equiv 0, x_3^2 \equiv 0, x_2x_3 \equiv 0, x_1x_3 \equiv 0$ and $x_1x_2 \equiv 0$. Hence, $\mathbf{C}_x/TQSWL_{G_0} \simeq \mathbf{C}_{x_1}$. According to Remark 2.10, the germ G is quasi swallowtail equivalent to $\pm x_2^2 \pm x_3^2 + \varphi_1(x_1)$ with $\varphi_1 \in \mathcal{M}_{x_1}^3$.

Similarly, we can show that

- the germ $H = \pm x_1^2 \pm x_3^2 + h_3(x)$, where $h_3 \in \mathcal{M}_x^3$ is quasi swallowtail equivalent to $\pm x_1^2 \pm x_2^2 + \varphi_2(x_2)$ with $\varphi_2 \in \mathcal{M}_{x_2}^3$, and
- the germ $P = \pm x_1^2 \pm x_2^2 + p_3(x)$, where $p_3 \in \mathcal{M}_x^3$ is quasi swallowtail equivalent to $\pm x_1^2 \pm x_2^2 + x_1\varphi_3(x_3) + \varphi_4(x_3)$, where $\varphi_3 \in \mathcal{M}_{x_3}^2$ and $\varphi_4 \in \mathcal{M}_{x_3}^3$. □

Lemma 3.5 *A function germ $f(x, y_1) = \pm x_1^2 \pm x_2^2 \pm x_3^2 + f_3(x, y_1)$ with $f_3 \in \mathcal{M}_{x, y_1}^3$, is quasi swallowtail equivalent to $\tilde{f}(x, y_1) = \pm x_1^2 \pm x_2^2 \pm x_3^2 + x_1\phi_1(y_1) + x_2\phi_2(y_1) + x_3\phi_3(y_1) + \phi_4(y_1)$ where $\phi_1, \phi_2, \phi_3 \in \mathcal{M}_{y_1}^2$ and $\phi_4 \in \mathcal{M}_{y_1}^3$.*

Proof. Consider the principal part $f_0 = \pm x_1^2 \pm x_2^2 \pm x_3^2$. Then, we get $\text{mod } TQSWL_{f_0} : x_1^2 \equiv 0, x_2^2 \equiv 0, x_3^2 \equiv 0, x_1x_2 \equiv 0, x_1x_3 \equiv 0$ and $x_2x_3 \equiv 0$. Hence, we have $\mathbf{C}_{x, y_1}/TQSWL_{f_0} \simeq \{x_1\varphi_1(y_1) + x_2\varphi_2(y_1) + x_3\varphi_3(y_1) + \varphi_4(y_1) : \varphi_1, \varphi_2, \varphi_3, \varphi_4 \in \mathbf{C}_{y_1}\}$. Due to the constraint in the lemma on f_3 , the claim of the lemma follows. □

3.1 Proof of the Main Theorem 3.1

Lemmas 3.3, 3.4 and 3.5 yield that all simple quasi swallowtail singularities are among the following germs:

1. $G_1 = \pm x_2^2 \pm x_3^2 + \varphi_1(x_1)$ where $\varphi_1 \in \mathcal{M}_{x_1}^3$.
2. $G_2 = \pm x_1^2 \pm x_3^2 + \varphi_2(x_2)$ where $\varphi_2 \in \mathcal{M}_{x_2}^3$.
3. $G_3 = \pm x_1^2 \pm x_2^2 + x_1\varphi_3(x_3) + \varphi_4(x_3)$ where $\varphi_3 \in \mathcal{M}_{x_3}^2$ and $\varphi_4 \in \mathcal{M}_{x_3}^3$.
4. $G_4 = \pm x_1^2 \pm x_2^2 \pm x_3^2 + x_1\phi_1(y_1) + x_2\phi_2(y_1) + x_3\phi_3(y_1) + \phi_4(y_1)$ where $\phi_1, \phi_2, \phi_3 \in \mathcal{M}_{y_1}^2$ and $\phi_4 \in \mathcal{M}_{y_1}^3$.

Using Lemma 2.9, one can easily prove the results below.

Consider the germ G_1 . Let $\varphi_1(x_1) = a_k x_1^k + \tilde{\varphi}_1(x_1)$, where $a_k \neq 0, k \geq 3$ and $\tilde{\varphi}_1 \in \mathcal{M}_{x_1}^{k+1}$. Then, G_1 is quasi swallowtail equivalent to the germ $\mathcal{Q}_k : \pm x_2^2 \pm x_3^2 \pm x_1^k$.

Similarly the germ G_2 can be reduced to one of the functions $\mathcal{T}_k : \pm x_1^2 \pm x_3^2 \pm x_2^k, k \geq 3$.

Next, consider the germ G_3 . If φ_4 contains a term $ax_3^3, a \neq 0$, then G_3 is quasi swallowtail equivalent to $\mathcal{D}_3 : \pm x_1^2 \pm x_2^2 \pm x_3^3$. Otherwise, in the most general case, G_2 is quasi swallowtail equivalent to a non-simple germ $\mathcal{D}_4 : \pm x_2^2 \pm (x_1 + \delta x_3^2)^2 \pm x_3^4, \delta \in \mathbb{R}$.

Finally, consider the germ $G_4 = \pm x_1^2 \pm x_2^2 \pm x_3^2 + x_1(a_2y_1^2 + a_3y_1^3 + a_4y_1^4 + \dots) + x_2(b_2y_1^2 + b_3y_1^3 + b_4y_1^4 + \dots) + x_3(c_2y_1^2 + c_3y_1^3 + c_4y_1^4 + \dots) + d_3y_1^3 + d_4y_1^4 + d_5y_1^5 + \dots$

- If $d_3 \neq 0$, then G_4 is quasi swallowtail equivalent to $\pm x_1^2 \pm x_2^2 \pm x_3^2 + y_1^3$, which can be written equivalently as $\mathcal{N}_{2,2,2,3} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm (x_3 + y_1^2)^2 + y_1^3$.
- If $d_3 = 0, a_2 \neq 0$ and $d_4 \neq \frac{a_2^2}{4}$, then G_4 is quasi swallowtail equivalent to $\pm x_1^2 \pm x_2^2 \pm x_3^2 + x_1y_1^2 \pm y_1^4$, which is also quasi swallowtail equivalent to $\mathcal{N}_{2,2,2,4} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm (x_3 + y_1^2)^2 \pm y_1^4$.

- If $d_3 = 0$, $a_2 \neq 0$ and $d_4 = \pm \frac{a_2^2}{4}$, then we get the classes $\mathcal{N}_{2,2,2,k} : \pm(x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm y_1^k$, where $k \geq 5$
- If $d_3 = a_2 = 0$, $b_2 \neq 0$ and $d_4 \neq \pm \frac{b_2^2}{4}$, then G_4 is quasi swallowtail equivalent to $\pm x_1^2 \pm x_2^2 \pm x_3^2 \pm x_2 y_1^2 \pm y_1^4$, which is equivalent to $\mathcal{N}_{3,2,2,4} : \pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^2)^2 \pm (x_3 + y_1^2)^2 \pm y_1^4$.
- If $d_3 = a_2 = 0$, $b_2 \neq 0$ and $d_4 = \pm \frac{b_2^2}{4}$, then we get a non-simple class $\mathcal{N}_{3,2,2,5} : \pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^2)^2 \pm (x_2 + y_1^2)^2 + \alpha y_1^5$ with $\alpha \in \mathbb{R} \setminus \{0\}$.
- If $d_3 = a_2 = b_2 = 0$, and $c_4 \neq 0$ and $d_4 \neq \pm \frac{c_2^2}{4}$, then we get the class $\pm x_1^2 \pm x_2^2 \pm x_3^2 \pm x_3 y_1^2 \pm y_1^4$, or, equivalently, the $\mathcal{N}_{3,3,2,4}$ class: $\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_3 + y_1^2)^2 \pm y_1^4$.
- If $d_3 = a_2 = b_2 = 0$, $c_2 \neq 0$ and $d_4 = \pm \frac{c_2^2}{4}$, then we get a non-simple class $\mathcal{N}_{3,3,2,5} : \pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_2 + y_1^2)^2 + \gamma y_1^5$ with $\gamma \in \mathbb{R} \setminus \{0\}$.
- If $d_3 = a_2 = b_2 = c_2 = 0$ and $d_4 \neq 0$, then we get the class $\pm x_1^2 \pm x_2^2 \pm x_3^2 \pm y_1^4$ or, equivalently, the $\mathcal{N}_{3,3,3,4}$ class: $\pm(x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_3 + y_1^3)^2 \pm y_1^4$.

This finishes the proof of the theorem.

4 BIFURCATION DIAGRAMS AND CAUSTICS OF SIMPLE QUASI SWALLOWTAIL SINGULARITIES

A quasi swallowtail miniversal deformation of a function germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ may be constructed in the standard way as

$$F(x, y, \lambda) = f(x, y) + \sum_{i=0}^{\tau-1} \lambda_i e_i(x, y), \tag{4}$$

where $e_0, \dots, e_{\tau-1} \in \mathbf{C}_{x,y}$ project to a basis of $\mathbf{C}_{x,y}/TQSWL_f$. We will use the notation F_λ for $F|_{\lambda=const}$, so that $F_0 = f$.

Definition 4.1 The *quasi swallowtail bifurcation diagram* of a function germ f is the set of all points λ in the base \mathbb{R}^τ of its quasi swallowtail miniversal deformation for which

- either the set $\{F_\lambda = 0\} \subset \mathbb{R}^n$ is singular,
- or a singularity of $\{F_\lambda = 0\}$ is on Γ_{swl} .

Respectively, this diagram consists of two components, W_1 and W_2 : $W_2 \subset W_1$, $\dim W_j = \tau - j$.

Now assume that $e_0 = 1$ in (4), and all the other e_i are from $\mathcal{M}_{x,y}$. Following the standard approach, we call the space $\mathbb{R}^{\tau-1}$ of the parameter $\lambda_1, \dots, \lambda_{\tau-1}$ the base of a *truncated quasi swallowtail miniversal deformation* of f .

Definition 4.2 Consider the projection map $\Pi : \mathbb{R}^\tau \rightarrow \mathbb{R}^{\tau-1}$ between the two bases, forgetting λ_0 . The *quasi swallowtail caustic* of a function f is a hypersurface in the base $\mathbb{R}^{\tau-1}$ which is a union of the Π -image Σ_1 of the singular points of the set $W_1 \subset \mathbb{R}^\tau$, and of the set $\Sigma_2 = \Pi(W_2)$.

Remark 4.3 In terms of Section 5 below, the component W_1 is the critical value set of the Lagrangian map of the manifold L , and W_2 is the image of the border Γ_{swl} .

The versal deformations listed below provide an explicit description of the bifurcation diagrams and caustics of simple quasi swallowtail singularities.

Proposition 4.4 *Quasi swallowtail miniversal deformations of simple quasi swallowtail classes are as follows:*

Singularity	Miniversal deformation	Restrictions
\mathcal{Q}_k	$\pm x_2^2 \pm x_3^2 \pm x_1^k + \sum_{i=0}^{k-1} \lambda_i x_1^i + \lambda_k x_2 + \lambda_{k+1} x_3$	$k \geq 2$
\mathcal{T}_k	$\pm x_1^2 \pm x_3^2 \pm x_2^k + \sum_{i=0}^{k-1} \lambda_i x_2^i + \lambda_k x_1 + \lambda_{k+1} x_1 + \lambda_{k+2} x_1 x_2$	$k \geq 3$
\mathcal{D}_3	$\pm x_1^2 \pm x_2^2 \pm x_3^3 + \lambda_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_3^2 + \lambda_5 x_1 x_3 + \lambda_6 x_2 x_3$	
$\mathcal{N}_{2,2,2,k}$	$\pm (x_1 + y_1^2)^2 \pm (x_2 + y_1^2)^2 \pm (x_3 + y_1^2)^2 \pm y_1^k + \sum_{i=0}^{k-2} \lambda_i y_1^i + \lambda_{k-1} x_1 + \lambda_k x_2 + \lambda_{k+1} x_3 + \lambda_{k+2} x_1 y_1 + \lambda_{k+3} x_2 y_1 + \lambda_{k+4} x_3 y_1$	$k \geq 3$
$\mathcal{N}_{3,2,2,4}$	$\pm (x_1 + y_1^3)^2 \pm (x_2 + y_1^2)^2 \pm (x_3 + y_1^2)^2 \pm y_1^4 + \lambda_0 + \lambda_1 y_1 + \lambda_2 y_1^2 + \lambda_3 x_1 + \lambda_4 x_2 + \lambda_5 x_3 + \lambda_6 x_1 y_1 + \lambda_7 x_2 y_1 + \lambda_8 x_3 y_1 + \lambda_9 x_1 y_1^2$	
$\mathcal{N}_{3,3,2,4}$	$\pm (x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_3 + y_1^2)^2 \pm y_1^4 + \lambda_0 + \lambda_1 y_1 + \lambda_2 y_1^2 + \lambda_3 x_1 + \lambda_4 x_2 + \lambda_5 x_3 + \lambda_6 x_1 y_1 + \lambda_7 x_2 y_1 + \lambda_8 x_3 y_1 + \lambda_9 x_1 y_1^2 + \lambda_{10} x_2 y_1^2$	
$\mathcal{N}_{3,3,3,4}$	$\pm (x_1 + y_1^3)^2 \pm (x_2 + y_1^3)^2 \pm (x_3 + y_1^3)^2 \pm y_1^4 \lambda_0 + \lambda_1 y_1 + \lambda_2 y_1^2 + \lambda_3 x_1 + \lambda_4 x_2 + \lambda_5 x_3 + \lambda_6 x_1 y_1 + \lambda_7 x_2 y_1 + \lambda_8 x_3 y_1 + \lambda_9 x_1 y_1^2 + \lambda_{10} x_2 y_1^2 + \lambda_{11} x_3 y_1^2$	

All simple quasi swallowtail singularities are the A_k singularities with respect to the standard right equivalence. So, the first component of the quasi swallowtail bifurcation diagram of a simple quasi swallowtail function is a product of a generalized swallowtail and $\mathbb{R}^{\tau-k}$. A similar observation is valid for the first components of the caustics. In particular, the bifurcation diagram of \mathcal{Q}_2 in \mathbb{R}^4 is a smooth hypersurface and a swallowtail on it. The \mathcal{Q}_2 caustic in \mathbb{R}^3 is a union of a smooth surface and a swallowtail.

5 APPLICATION TO LAGRANGIAN BORDER SINGULARITIES

Standard notions and basic definitions concerning Lagrangian singularities can be found in [8].

Singularities of Lagrangian projections (mappings) are essentially the singularities of their generating families treated as families of functions depending on parameters and considered up to the right equivalence depending on parameters and addition of functions in parameters. In particular, the caustic $\Sigma(L)$ of a Lagrangian projection of a Lagrangian submanifold L coincides with the set of values of the parameters λ of the generating family $F(w, \lambda)$ for which the family member F_λ has a non-Morse critical point.

Stability of a Lagrangian projection with respect to symplectomorphisms preserving the fibration structure corresponds to the versality of the generating family with respect to the \mathcal{R}_+ -equivalence group.

Consider the standard symplectic space $M = T^*\mathbb{R}^n$ with coordinates q on the base \mathbb{R}^n and dual coordinates p on the fibers of the Lagrangian projection $\pi : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Locally any Lagrangian submanifold L^n in an ambient symplectic space M is determined by a generating family of functions $F(w, q)$ in variables $w \in \mathbb{R}^m$ and parameters $q \in \mathbb{R}^n$ according to the standard formula:

$$L = \left\{ (p, q) \in \mathbb{R}^n \times \mathbb{R}^n : \exists w \in \mathbb{R}^m, \frac{\partial F}{\partial w_i} = 0, p = \frac{\partial F}{\partial q} \right\},$$

provided that the Morse non-degeneracy condition [the matrix $\begin{pmatrix} \frac{\partial^2 F}{\partial w_i \partial w_j} & \frac{\partial^2 F}{\partial w_i \partial q_j} \end{pmatrix}$ has rank n] holds. The condition guarantees L being a smooth manifold.

Definition 5.1 [8] Two family germs $F_i(w, q)$, $w \in \mathbb{R}^m$, $q \in \mathbb{R}^n$, $i = 1, 2$, at the origin are called \mathcal{R}_+ -equivalent if there exists a diffeomorphism $\Phi : (w, q) \mapsto (W(w, q), Q(q))$ and a smooth function Θ of the parameters q such that $F_2(w, q) = (F_1 \circ \Phi)(w, q) + \Theta(q)$.

Following applications of quasi boundary, quasi corner and quasi cusp equivalence relations considered in [2, 3, 4], we introduce

Definition 5.2 [4] A pair (L, Γ) consisting of a Lagrangian submanifold L^n in an ambient symplectic space M and an $(n-1)$ -dimensional isotropic variety $\Gamma \subset L$ is called a *Lagrangian submanifold with a border* Γ .

Definition 5.3 [4] Lagrangian projections of two Lagrangian submanifolds with borders (L_i, Γ_i) , $i = 1, 2$, are *Lagrange equivalent* if there exists a symplectomorphism of the ambient space M which preserves the π -bundle structure and sends one pair to the other.

The notions of stability and simplicity of Lagrangian submanifolds with borders with respect to this Lagrangian equivalence are straightforward.

Up to a Lagrange equivalence we may assume that in a vicinity of a base point the tangent space to L has a regular projection onto the fiber of π and the coordinates p can be taken as coordinates w on the fibers of the source space of the generating family.

Generating family is defined up to \mathcal{R}_+ -equivalence. So having two Lagrange equivalent pairs (L_i, Γ_i) we can choose a generating family for one of them in coordinates p, q and the generating family for the second pair in transformed coordinates $\tilde{P}(p)$ so that the projection of Γ_1 to p -coordinate subspace coincides with the projection of Γ_2 to the \tilde{P} -coordinate subspace.

Assume that the Γ_i are borders, $i = 1, 2$. Rename the coordinates p by w and q by λ . Let $g_i(w) = 0$ be the equation of the border Γ_i , $i = 1, 2$.

Now we have generating families $F_i(w, \lambda)$ for both submanifolds such that the critical points of F_i with respect to variables w at the set $\{g_i(w) = 0\}$ correspond to the border Γ_i .

Hence, the Lagrange equivalence of pairs (L_i, Γ_i) , $i = 1, 2$, gives rise to an equivalence of the generating families F_i which is a pseudo border equivalence and addition with a function in parameters.

Moreover the following holds.

Proposition 5.4 *Let (L_t, Γ_t) , $t \in [0, 1]$, be a family of equivalent pairs of Lagrangian submanifolds with a cylinder over a swallowtail stratum. Then the respective generating families are **quasi swallowtail** equivalent up to addition of functions depending on parameters.*

The above equivalence of generating families will be called the *quasi swallowtail +-equivalence*.

The last proposition and the classification of simple quasi swallowtail singularities imply the following theorem.

Theorem 5.5 *1. A germ (L, Γ) is stable if and only if its arbitrary generating family is quasi border +-versal, that is, versal with respect to the quasi border equivalence and addition of functions in parameters.*

2. Any stable and simple projection of a Lagrangian submanifold with a cylinder over a swallowtail stratum is symplectically equivalent to the projection determined by a generating family which is a quasi swallowtail +-versal deformation of one of the classes from Theorem 3.1.

Proof. Suppose that a germ (L_0, Γ_0) is stable. Then any germ $(\tilde{L}, \tilde{\Gamma})$ close (L_0, Γ_0) is Lagrange equivalent to it.

Assume we have a family (L_t, Γ_t) of deformations of (L_0, Γ_0) , with $t \in [0, 1]$. Also assume that there is a family of diffeomorphism $\theta_t : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ which preserves Lagrange fibration $\pi : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$, $(p, q) \mapsto q$ and the standard symplectic form ω , and maps (L_t, Γ_t) to (L_0, Γ_0) .

Consider families depending on t of respective generating families $G_t(w, q)$ of (L_t, Γ_t) with $t \in [0, 1]$ and G_0 being a generating family of the pair (L_0, Γ_0) . By Proposition 5.4, all the G_t are quasi swallowtail +-equivalent. Thus, there exist a family of diffeomorphisms $\Phi_t : (w, q) \mapsto (\tilde{w}_t(w, q), Q_t(q))$ and a family Ψ_t of smooth functions of the parameters q such that: $G_t \circ \Phi_t = G_0 + \Psi_t$, and the critical points sets $\{\frac{\partial G_t}{\partial w} = 0\}$ correspond to the Lagrangian submanifolds L_t . This yields, in particular, that G_0 is versal with respect to quasi border +-equivalence.

By reversing the previous argument we prove the converse claim.

The second part of the theorem is a consequence of the classification of function germs with respect to the quasi swallowtail equivalence. □

6 CONCLUSION

The further study of similar non-standard equivalence relation and its comparison with standard one will give extra information on the nature of singularity classes in various optimization problems and problems in variations theory with constraints, on the adjacencies of respective singularity classes and on their invariants.

ACKNOWLEDGEMENTS: I am deeply grateful to Vladimir Zakalyukin who introduced me to the idea of quasi border equivalence relation. I am also thankful to Victor Goryunov for a helpful discussion and his highly useful comments on the first draft of the paper.

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