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Sum Perfect Square Labeling of Graphs

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Abstract: *Let $G=(V,E)$ be a (p,q) graph and let $f:V(G)\rightarrow\{0,1,2,......., p-1\}$ be a bijection. We define $f^*:E(G)\rightarrow N$ by $f^*(uv)=[f(u)]^2+[f(v)]^2+2f(u)\bullet f(v)$. If f^* is injective, then f is called sum perfect square labeling. A graph which admits sum perfect square labeling is called sum perfect square graph. Here we have focused on the graphs, whose edges can be labeled by a perfect square number only. In this paper we have initiated study of graphs which are sum perfect square.*

Keywords: *Sum perfect square graphs.*

AMS Subject classification number: 05C78

1. INTRODUCTION

In our new defined labeling, the actual idea is to label all the edges of a graph by a perfect square number only. We consider simple, finite, undirected graph G (with p vertices and q edges) throughout all this paper. The vertex set and edge set of graph G are denoted by $V(G)$ and $E(G)$ respectively. In this paper P_n denotes path with n vertices, $T_{m,n}$ denotes tadpole with $m+n$ vertices, $K_{1,n}$ denotes star graph with n+1vertices. For all other terminologies and notations we follow Harary[1].

Definition 1.1 : A chord of a cycle is an edge joining two non-adjacent vertices of cycle.

Definition 1.2: Two chords of a cycle are said to be twin chords if they form a triangle with an edge of the cycle. For positive integers n and p with $3 \leq p \leq n-2$, $C_{n,p}$ is the graph consisting of a cycle C_n with a pair of twin chords with which the edges of C_n form cycles C_p, C_3 and C_{n+1-p} without chords.

Definition 1.3: The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G)\cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and other is an edge incident with it.

Definition 1.4: Let G_1 *and* G_2 be two graphs such that $V(G_1)\cap V(G_2)=\varnothing$. The join of G_1 *and* G_2 is denoted by G_1+G_2. It is the graph with $V(G_1+G_2)=V(G_1)\cup V(G_2)$ and $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup J$, where $J=\{uv / u\in V(G_1)\, and\, v\in V(G_2)\}$.

Definition 1.5: Let $G=(p,q)$ be a graph. A bijection $f:V(G)\rightarrow\{0,1,2,......., p-1\}$ is called sum perfect square labeling of G, if the induced function $f^*:E(G)\rightarrow N$ by $f^*(uv)=[f(u)]^2+[f(v)]^2+2f(u)\bullet f(v)$ is injective, for all $u,v\in V(G)$. A graph which admits sum perfect square labeling is called sum perfect square graph.

2. LITERATURE SURVEY

V Ajitha, S. Arumugam and K. Germina[7] have initiated the study of Square Sum Graphs in 2009. Their work is closely related to Diophantine kinds of equations. The authors have shown several graphs which are Square sum. K. Germina et al[4] also proved several more graphs to be Square

Sum graphs. J. Shiama [4] defined Square difference graphs and also find several graphs, which are Square difference. Above work motivates and leads us to define a new labeling, we say sum perfect square labeling. We say a graph to be sum perfect square if it admits sum perfect square labeling. Due to new defined labeling, we find the existence of the graphs, whose edges can be labeled by a perfect square number only.

3. Main Results

Lemma 3.1: P_n is sum perfect square graph, for all $n \in N$, $n \geq 2$.

Proof: Let $\{v_1, v_2, ..., v_n\}$ be the successive vertices of path P_n. We define $f: V(P_n) \to \{0, 1, 2, ..., n-1\}$ as $f(v_i) = i-1, 1 \leq i \leq n$.

Injectivity: As function f is increasing in terms of i, we get $f(v_i) < f(v_{i+1})$ and so $f^*: (E(G)) \to N$ is injective. Hence P_n is sum perfect square graph, for all $n \in N$, $n \geq 2$.

Lemma 3.2: Star $K_{1,n}$ is sum perfect square graph, for all $n \in N$.

Proof: Let $\{v_0, v_1, ..., v_n\}$ be the successive vertices of star $K_{1,n}$, where v_0 is the apex vertex. We define $f: V(K_{1,n}) \to \{0, 1, 2, ..., n\}$ as $f(v_0) = 0$ and $f(v_i) = i, 1 \leq i \leq n$.

Injectivity: As function f is increasing in terms of

i *for any* $i, m \in N$, *with* $i < m$, *we have* $f(v_i) < f(v_m)$, and so $f^*(v_0 v_i) < f^*(v_0 v_m), 1 \leq i, m \leq n.$

So $f^*: (E(G)) \to N$ is injective. Hence $K_{1,n}$ is sum perfect square graph, for all $n \in N$.

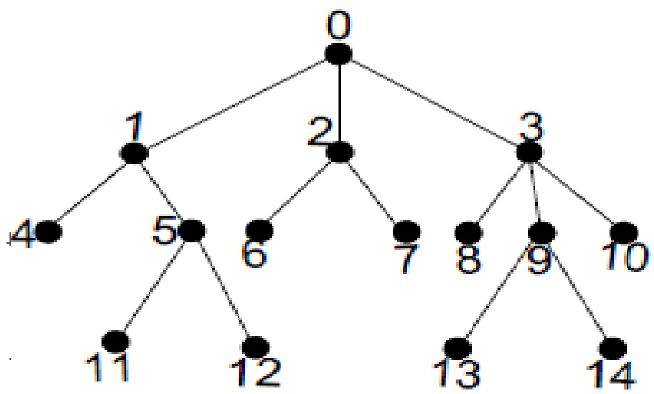
Theorem 3.1: Every tree is sum perfect square graph.

Proof : Let $\{v_0, v_1, ..., v_n\}$ be the successive vertices of a tree T. Let us consider T as a rooted tree. Let v_0 be the root vertex of T. Clearly v_0 is at vertex level 0. Let $\{v_1, v_2, ..., v_k\}$ be the successive vertices at level 1, $1 \leq k \leq n$ $(k \in N)$. $\{v_{k+1}, v_{k+2}, ..., v_t\}$ be the successive vertices at level 2, $k+1 \leq t \leq n$ $(t \in N)$ and so on. We define $f: V(T) \to \{0, 1, 2, ..., n\}$ by $f(v_i) = i$, $0 \leq i \leq n$.

Injectivity: At each consecutive level l_m *and* l_{m+1}, $m \in N \cup \{0\}$, *we get* $f(v_i) < f(v_j)$,

for all i *and* j, where vertex v_i is at level l_m and vertex v_j is at level l_{m+1}. So $f^*: (E(T)) \to N$ is injective. Hence tree is sum perfect square graph.

- An illustration of tree with 15 vertices is also provided below, for better understanding of defined labeling pattern.



Theorem 3.2 : C_n is sum perfect square graph, for all $n \geq 3$.

Proof : Let $\{v_1, v_2, ..., v_n\}$ be the successive vertices of cycle C_n. We define $f: V(C_n) \to \{0, 1, 2, ..., n-1\}$ as follows:

$$f(v_i) = \begin{cases} 2i-2,; & 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 2(n-i)+1; & \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \end{cases}.$$

Injectivity :

For $1 \le i \le \left\lceil \frac{n}{2} \right\rceil$ since f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2}), 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1$. Moreover for $\left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n$, as f is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) > f^*(v_{i+1} v_{i+2}), \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n.$ We also note that $f^*(v_1 v_n) = 1,$ which is the smallest edge label in the graph.

Further $f^*(v_{\left\lceil \frac{n}{2} \right\rceil} v_{\left\lceil \frac{n}{2} \right\rceil + 1}) = \begin{cases} (2n-4)^2, & if\ n\ is\ even, \\ (2n-3)^2, & if\ n\ is\ odd \end{cases}$ which is the highest edge label in the graph. Now $f(v_t)$ is even for each t, $1 \le t \le \left\lceil \frac{n}{2} \right\rceil$ and $f(v_k)$ is odd for each k, $\left\lceil \frac{n}{2} \right\rceil + 1 \le k \le n-1,$ we get $f^*(v_t v_{t+1}) \ne f^*(v_k v_{k+1}).$ So $f^*: (E(C_n)) \to N$ is injective. Hence C_n is sum perfect square graph, for all $n \ge 3$.

Theorem 3.3: Cycle C_n with one chord is sum perfect square graph, for all $n \ge 4$.

Proof : Let G be the cycle with one chord. Let $\{v_1, v_2, ..., v_n\}$ be the successive vertices of cycle C_n and $e = v_2 v_n$ be the chord of cycle C_n. The vertices v_1, v_2 *and* v_n form a triangle in C_n with chord e. We define $f : V(C_n) \to \{0, 1, 2, ..., n-1\}$ as follows:

$$f(v_i) = \begin{cases} 2i-2,; & 1 \le i \le \left\lceil \frac{n}{2} \right\rceil \\ 2(n-i)+1; & \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n \end{cases}.$$

Injectivity:

The chord $e = v_2 v_n$ is labeled by 9, which is different from all other edge labels, as this edge label can be induced only by the vertex labels 0,3 and 1,2, but after label the vertex v_1 by 1 and v_n by 2, the pair 0,3 will never meet each other. For $1 \le i \le \left\lceil \frac{n}{2} \right\rceil$, since f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2}), 1 \le i \le \left\lceil \frac{n}{2} \right\rceil - 1$. Moreover for $\left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n$, as f is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) > f^*(v_{i+1} v_{i+2}), \left\lceil \frac{n}{2} \right\rceil + 1 \le i \le n.$ Also $f^*(v_1 v_n) = 1,$ which is unique and smallest edge label among all the edge labels in the graph.

Further $f^*(v_{\left\lceil \frac{n}{2} \right\rceil} v_{\left\lceil \frac{n}{2} \right\rceil + 1}) = \begin{cases} (2n-4)^2, & if\ n\ is\ even, \\ (2n-3)^2, & if\ n\ is\ odd, \end{cases}$ which is the highest edge label and so different from all other edge labels in the graph. Now $f(v_t)$ is even for each t, $1 \le t \le \left\lceil \frac{n}{2} \right\rceil - 1$ and $f(v_k)$ is odd for each k, $\left\lceil \frac{n}{2} \right\rceil + 1 \le k \le n-1,$ we get $f^*(v_t v_{t+1}) \ne f^*(v_k v_{k+1}).$ So all the edge labels are distinct. Hence C_n with one chord is sum perfect square graph, for all $n \ge 4$.

Theorem 3.4: Cycle C_n with twin chords is sum perfect square graph, for all $n \geq 5$.

Proof : Let G be the cycle with twin chords. Let $\{v_1, v_2, ..., v_n\}$ be the successive vertices of cycle C_n and $e_1 = v_2v_n$ and $e_2 = v_3v_n$ be the two chords of cycle C_n.

We define $f : V(C_n) \rightarrow \{0,1,2,.....,n-1\}$ as follows:

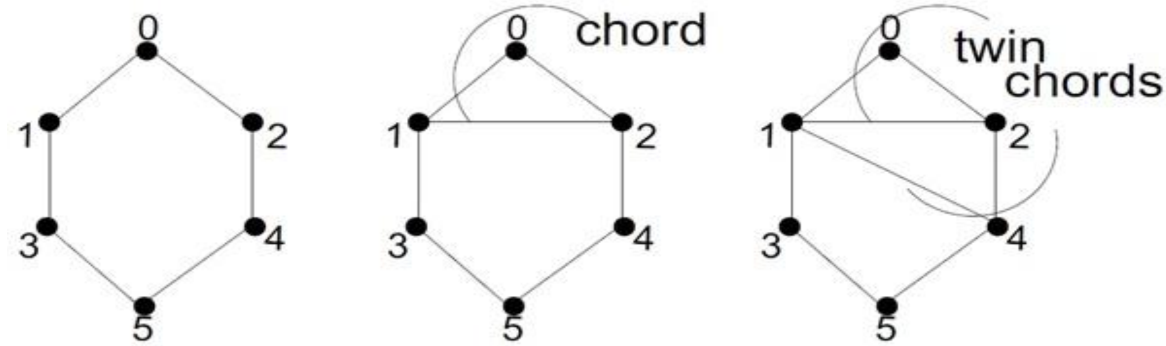
$$f(v_i) = \begin{cases} 2i-2,; & 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\ 2(n-i)+1; & \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n \end{cases}.$$

Injectivity:

The chord $e_1 = v_2v_n$ is labeled by 9, which is different from all other edge labels, as this edge label can be induced only by the vertex labels 0,3 and 1,2, but after label the vertex v_1 by 1 and v_n by 2, the pair 0,3 will never meet each other, similar argument is valid for $e_2 = v_3v_n$, which is labeled by the label 25. For $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, since f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f^*(v_iv_{i+1}) < f^*(v_{i+1}v_{i+2}), 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$. Moreover for $\left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n$, as f Is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f^*(v_iv_{i+1}) > f^*(v_{i+1}v_{i+2}), \left\lceil \frac{n}{2} \right\rceil + 1 \leq i \leq n.$ Also $f^*(v_1v_n) = 1$, which is unique and smallest edge label among all the edge labels in the graph.

Further $f^*(v_{\left\lceil \frac{n}{2} \right\rceil} v_{\left\lceil \frac{n}{2} \right\rceil + 1}) = \begin{cases} (2n-4)^2, & if\ n\, is\, even, \\ (2n-3)^2, & if\ n\, is\, odd, \end{cases}$ which is the highest edge label and so different from all other edge labels in the graph. Now $f(v_t)$ is even for each t, $1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 1$ and $f(v_k)$ is odd for each k, $\left\lceil \frac{n}{2} \right\rceil + 1 \leq k \leq n-1$, we get $f^*(v_tv_{t+1}) \neq f^*(v_kv_{k+1})$. So all the edge labels are distinct. Hence C_n with twin chords is sum perfect square graph, for all $n \geq 5$.

- Following figure justifies the labeling pattern, which we have defined in Theorem 2, 3 and 4.



Theorem 3.5 : K_n is sum perfect square graph, for $n < 4$.

Proof : For $n = 1$ *and* 2, the graphs are special cases of tree and according to Theorem 3.1, K_1 and K_2 are sum perfect square graphs. For $n = 3$, K_3 is the cycle C_3, and hence as per Theorem 3.2, K_3 is sum perfect square graph. For $n = 4$, when we label the vertices of K_4, we come across the pair 1, 2 and 0, 3, for which we get the same induced edge labels. Hence K_4 is not sum perfect square graph. For all $n \geq 4$, as $K_4 \subset K_5 \subset .... \subset K_n$, and so K_n is not sum perfect square graph, for all $n \geq 4$.

Theorem 3.6: Tadpole $T_{m,n}$ is sum perfect square gaph, for all positive integers m *and* n.

Proof: Let $\{v_1, v_2, ..., v_m\}$ be the successive vertices of cycle C_m and let $\{v_{m+1}, v_{m+2}, ..., v_{m+n}\}$ be the successive vertices of path P_n in tadpole $T_{m,n}$. Let $e = v_m v_{m+1}$ be the bridge in tadpole $T_{m,n}$, where v_m is the vertex with the highest label corresponding to cycle C_n and v_{m+1} is the vertex with the smallest label corresponding to path P_n.

We define $f : V(T_{m,n}) \rightarrow \{0,1,2,.....,m+n-1\}$ as follows:

Case 1: m is even.

$$f(v_i) = \begin{cases} m - 2i, \ 1 \le i \le \left\lceil \frac{m}{2} \right\rceil, \\ 2i - m - 1, \ \left\lceil \frac{m}{2} \right\rceil + 1 \le i \le m, \\ i - 1, \ m+1 \le i \le m+n \end{cases}$$

Injectivity:

For $1 \le i \le \left\lceil \frac{m}{2} \right\rceil$, since f is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) > f^*(v_{i+1} v_{i+2}), \ 1 \le i \le \left\lceil \frac{m}{2} \right\rceil - 1$. For $\left\lceil \frac{m}{2} \right\rceil + 1 \le i \le m+n$, as f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2}), \left\lceil \frac{m}{2} \right\rceil + 1 \le i \le m+n$. Moreover for Also $f^*(v_{\lceil \frac{m}{2} \rceil} v_{\lceil \frac{m}{2} \rceil + 1}) = 1$, which is the smallest edge label among all the edge labels in this graph. Further $f^*(v_1 v_m) = (2m-3)^2$, is the highest edge label in cycle C_m and according to the pattern which we defined, for $m+1 \le i \le m+n$, as f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2})$. Now $f(v_t)$ is even for each t, $1 \le t \le \left\lceil \frac{m}{2} \right\rceil$ and $f(v_k)$ is odd for each k, $\left\lceil \frac{m}{2} \right\rceil + 1 \le k \le m$, we get $f^*(v_t v_{t+1}) \ne f^*(v_k v_{k+1})$. So all the edge labels are distinct. Hence $f^* : (E(T_{m,n})) \rightarrow N$ is injective in this case.

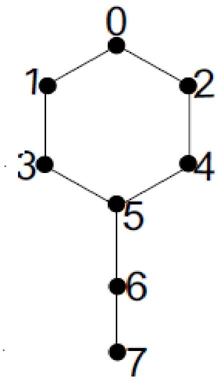
Case 2: m is odd.

$$f(v_i) = \begin{cases} m - 2i - 1, \ 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor, \\ 2i - m - 1, \ \left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m-1, \\ i - 1, \ m+1 \le i \le m+n \end{cases}$$

Injectivity:

For $1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor$, since f is decreasing in terms of i, $f(v_i) > f(v_{i+1}) > f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) > f^*(v_{i+1} v_{i+2})$, $1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor - 1$. For $\left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m+n$, as f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2})$, $\left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m+n$. Moreover for Also $f^*(v_{\left\lfloor \frac{m}{2} \right\rfloor} v_{\left\lfloor \frac{m}{2} \right\rfloor + 1}) = 1$, which is the smallest edge label among all the edge labels in this graph. Further $f^*(v_1 v_m) = (2m-3)^2$, is the highest edge label in cycle C_m and according to the pattern which we defined, for $m+1 \le i \le m+n$, as f is increasing in terms of i, $f(v_i) < f(v_{i+1}) < f(v_{i+2})$ and so we get $f^*(v_i v_{i+1}) < f^*(v_{i+1} v_{i+2})$. Now $f(v_t)$ is even for each t, $1 \le t \le \left\lfloor \frac{m}{2} \right\rfloor$ and $f(v_k)$ is odd for each k, $\left\lfloor \frac{m}{2} \right\rfloor + 1 \le k \le m$, we get $f^*(v_t v_{t+1}) \ne f^*(v_k v_{k+1})$. So all the edge labels are distinct. Hence $f^* : (E(T_{m,n})) \to N$ is injective in this case. Tadpole $T_{m,n}$ is sum perfect square graph, for all positive integers m *and* n.

- The following figure indicates the labelling pattern defined in Theorem no. 6



4. CONJECTURE

Every odd graph (i.e. all the vertices are having odd degree) G, in which all the vertices v_i are such that $d(v_i) \ge 3$ then G is not a sum perfect square graph.

5. CONCLUSION

The labeling introduced in this paper will open new notion of study of some families of several graphs, in which all the edges must be labeled by a distinct perfect square integer. Related to new introduced labeling, several new theorems have been proved in this paper. At the end a conjecture have been put, which can be treated as an open problem.

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