International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 4, Issue 1, January 2016, PP 46-52
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

A Study of Double Euler Summability Method of Fourier Series and its Conjugate Series

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Abstract: *In this paper, the product of Euler means is taken up to study the double summability of Fourier series and its allied series. We established two new theorems on* $(E,1)(E,1)$ *product summability of Fourier series and its conjugate Fourier series.*

Keywords: (E,q) *summability,* $(E,1)(E,1)$ *summability.*

1. Definition and Notation

Let $f(x)$ be a 2π periodic function and Lebesgue integrable over $(-\pi,\pi)$. The Fourier series of $f(x)$ at any point x is given by

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\equiv\sum_{n=1}^{\infty}A_n(x) \tag{1.1}$$

The conjugate series of Fourier series is given by

$$\sum_{n=1}^{\infty}(b_n\cos nx-a_n\sin nx)\equiv\sum_{n=1}^{\infty}B_n(x) \tag{1.2}$$

We shall use the following notation,

$$\phi(t)=f(x+t)+f(x-t)-2s$$

$$\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$$

$$K_n(t)=\frac{1}{2^{n+k+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\sin(v+1/2)t}{\sin t/2}\right\}$$

$$\tilde{K}_n(t)=\frac{1}{2^{n+k+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\left\{\sum_{v=0}^{k}\binom{k}{v}\frac{\cos(v+1/2)t}{\sin t/2}\right\}$$

and

$$\tau=\left[\frac{1}{t}\right]\text{, where } \tau \text{ denotes the greatest integer not greater that } \frac{1}{t}$$

Let $\sum_{n=0}^{\infty}u_n$ be a given infinite series with sequence of its n^{th} partial sum $\{s_n\}$. The $(E,1)$ transform is defined as the n^{th} partial sum of $(E,1)$ summability and is given by

$$t_n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k \text{, then}$$

the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite no. s by $(E,1)(E,1)$ summability method if

$$t_n^{(E,1)(E,1)} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{1}{2^k}\sum_{v=0}^{k}\binom{k}{v}s_v\right\} \to s \text{, as } n \to \infty$$

2. INTRODUCTION

In the field of summability of Fourier series & its allied series, the product summability $(E,q)(X)$, $(X)(E,q)$ or $|E,q|$ have been studied by a number of researchers like, Mohanty, R. and Mohapatra, S. (1968), Kwee, B. (1972), [2]Chandra, P. (1977), [1]Chandra, P. and Dikshit, G.D. (1981), Sachan, M.P. (1983), Bhagwat, Purnima (1987), Nigam, H.K. and Sharma, Ajay (2006), Lal, S., Singh, H.P., Tiwari, [8]Sandeep kumar, and Bariwal, Chandrashekhar (2010), [3]Dhakal, Binod Prasad (2011), Rathore, H.L. and Shrivastava, U.K. (2012), Nigam, H.K. and Sharma, K. (2012,2013), Sinha, Santosh kumar & Shrivastava, U.K. (2014), Mishra, V.N. and Sonavane, Vaishali (2015) and many more, under various type of criteria & conditions. After this, so many results established on double factorable summability of double Fourier series, the methods of $(C,1,1)$, $(H,1,1)$&(N, p_m, q_n).But yet, no result found on double Euler summability of Fourier series & its allied series. Under a general condition, hear we have established two new theorems on $(E,1)(E,1)$ product summability of Fourier series and its Conjugate series.

3. MAIN THEOREM

Theorem 1: Let $\{p_n\}$be a positive, monotonic, non-increasing sequence of real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{, as } n \to \infty$$

If

$$\Phi(t) = \int_0^t |\phi(u)| du = o\left[\frac{t}{\alpha(1/t)}\right] \text{, as } t \to +0 \tag{3.1}$$

where $\alpha(t)$is positive, non-increasing function of t and $\alpha(n) \to \infty$, as $n \to \infty$. Then the Fourier series (1.1) is summable $(E,1)(E,1)$ to $f(x)$ at pt $t = x$.

Theorem 2: Let $\{p_n\}$be a positive, monotonic, non-increasing sequence of real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{, as } n \to \infty$$

If

$$\Psi(t) = \int_0^t |\psi(u)| du = o\left[\frac{t}{\alpha(1/t)}\right] \text{, as } t \to +0 \tag{3.2}$$

where $\alpha(t)$is positive, non-increasing function of t, then the Conjugate Fourier series (1.2) is summable to

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} \psi(t)\cot\left(\frac{t}{2}\right)dt$$

at every pt, where this integral exists.

4. LEMMAS

Lemma 1: For $0 \le t \le 1/n$; $\sin(t/2) \ge t/2$; $\sin nt \le n\sin t$; $|\cos nt| \le 1$

Proof: $$|K_n(t)| \le \frac{1}{2^{n+k+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\frac{\sin(v+1/2)t}{\sin t/2}\right|$$

$$\le \frac{1}{2^{n+k+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\frac{(2v+1)\sin t/2}{\sin t/2}\right|$$

$$\le \frac{1}{2^{n+k+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}(2k+1)\sum_{v=0}^{k}\binom{k}{v}\right|$$

$$= \frac{1}{2^{n+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}(2k+1)\right|$$

$$= \frac{1}{2^{n+1}\pi}(2n+1)\sum_{k=0}^{n}\binom{n}{k}$$

$$= \frac{(2n+1)}{2\pi}$$

$$= O(n)$$

Lemma 2: For $1/n \le t \le \pi$; $\sin(t/2) \ge t/2$;

Proof: $$|K_n(t)| \le \frac{1}{2^{n+k+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\frac{\sin(v+1/2)t}{\sin t/2}\right|$$

$$= \frac{1}{2^{n+k}\pi t}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\right]$$

$$= \frac{1}{2^{n}\pi t}\left[\sum_{k=0}^{n}\binom{n}{k}\right]$$

$$= \frac{1}{\pi t}$$

$$= O\left(\frac{1}{t}\right)$$

Lemma 3: For $0 \le t \le 1/n, \sin(t/2) \ge t/2$; $|\cos nt| \le 1$

Proof: $$|\tilde{K}_n(t)| \le \frac{1}{2^{n+k+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\frac{\cos(v+1/2)t}{\sin t/2}\right|$$

$$= \frac{1}{2^{n+k+1}\pi}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\frac{|\cos(v+1/2)|}{|\sin t/2|}\right]$$

$$= \frac{1}{2^{n+k}\pi t}\left[\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\right]$$

$$= \frac{1}{2^{n}\pi t}\sum_{k=0}^{n}\binom{n}{k}$$

$$= \frac{1}{\pi t}$$

$$= O\left(\frac{1}{t}\right)$$

Lemma 4: For $1/n \le t \le \pi$, $\sin(t/2) \ge t/2$

Proof: $$\left|\widetilde{K}_n(t)\right| \le \frac{1}{2^{n+k+1}\pi}\left|\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\frac{\cos(v+1/2)t}{\sin t/2}\right|$$

$$\le \frac{1}{2^{n+k}\pi t}\left|\sum_{k=0}^{n}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}\cos(v+1/2)t\right|$$

$$\le \frac{1}{2^{n+k}\pi t}\left|\sum_{k=0}^{n}\binom{n}{k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right\}\right|\left|e^{it/2}\right|$$

$$\le \frac{1}{2^{n+k}\pi t}\left|\sum_{k=0}^{n}\binom{n}{k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right\}\right|$$

$$\le \frac{1}{2^{n+k}\pi t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right\}\right| + \frac{1}{2^{n+k}\pi t}\left|\sum_{k=\tau}^{n}\binom{n}{k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right\}\right|$$

$$= k_1 + k_2$$

$$|k_1| = \frac{1}{2^{n+k}\pi t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right\}\right|$$

$$\le \frac{1}{2^{n+k}\pi t}\left|\sum_{k=0}^{\tau-1}\binom{n}{k}\left\{\sum_{v=0}^{k}\binom{k}{v}\right\}\right|\left|e^{ivt}\right|$$

$$\le \frac{1}{2^{n+k}\pi t}\sum_{k=0}^{\tau-1}\binom{n}{k}\sum_{v=0}^{k}\binom{k}{v}$$

$$= \frac{1}{2^{n}\pi t}\sum_{k=0}^{\tau-1}\binom{n}{k}$$

$$= O\left(\frac{1}{t}\right)$$

Now,

$$|k_2| \le \frac{1}{2^{n+k}\pi t}\left|\sum_{k=\tau}^{n}\binom{n}{k}\mathrm{Re}\left\{\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right\}\right|$$

$$\leq \frac{1}{2^{n+k}\pi t}\sum_{k=\tau}^{n}\binom{n}{k}\max_{0\leq m\leq k}\left|\sum_{v=0}^{k}\binom{k}{v}e^{ivt}\right|$$
$$= O\left(\frac{1}{t}\right)$$

5. Proof

Proof of Theorem 1: We have to show, under the hypothesis of the theorem, that

$$\int_0^\pi \phi(t)K_n(t)dt = o(1) \text{ , as } n\to\infty$$

we set, for $0<\delta<\pi$,

$$t_n^{(E,1)(E,1)} - f(x) = \int_0^\pi \phi(t)K_n(t)dt$$
$$= \left[\int_0^{1/n} + \int_{1/n}^{\delta} + \int_\delta^\pi\right]\phi(t)K_n(t)dt$$
$$= I_1 + I_2 + I_3 \text{, say} \qquad (5.1)$$

Now, let

$$|I_1| = \int_0^{1/n}|\phi(t)||K_n(t)|dt$$
$$= O(n)\left[\int_0^{1/n}|\phi(t)|dt\right] \text{ by Lemma 1}$$
$$= O(n)\left[o\left\{\frac{1}{n\alpha(n)}\right\}\right]$$
$$= o\left\{\frac{1}{\alpha(n)}\right\}$$
$$= o(1), \text{ as } n\to\infty \qquad (5.2)$$

$$|I_2| = \int_{1/n}^{\delta}|\phi(t)||K_n(t)|dt$$
$$= O\left[\int_{1/n}^{\delta}|\phi(t)|\left(\frac{1}{t}\right)dt\right] \text{ by Lemma 2}$$
$$= O\left[\left\{\frac{1}{t}\Phi(t)\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta}\frac{1}{t^2}\Phi(t)dt\right]$$
$$= O\left[o\left\{\frac{1}{\alpha(1/t)}\right\}_{1/n}^{\delta} + \int_{1/n}^{\delta}o\left\{\frac{1}{t\alpha(1/t)}\right\}dt\right]$$
$$= O\left[o\left\{\frac{1}{\alpha(n)}\right\} + \int_{1/\delta}^{n}o\left\{\frac{1}{u\alpha(u)}\right\}du\right]$$
$$= o\left\{\frac{1}{\alpha(n)}\right\} + o\left\{\frac{1}{n\alpha(n)}\right\}\int_{1/\delta}^{n}du$$

Using second-mean value theorem for the integral in the second term as $\alpha(n)$ is monotonic

$$= o(1) + o(1) \text{ as } n \to \infty$$

$$= o(1) \text{ as } n \to \infty \tag{5.3}$$

Finally,

$$|I_3| = \int_\delta^\pi |\phi(t)||K_n(t)|dt = o(1) \text{ as } n \to \infty \tag{5.4}$$

By using Riemann-Lebesgue theorem and regularity condition of summability.

Combining (5.2), (5.3) and (5.4) we have

$$t_n^{(E,1)(E,1)} - f(x) = o(1) \text{ , as } n \to \infty$$

This completes the proof of theorem 1.

Proof of Theorem 2:

$$\tilde{t}_n^{(E,1)(E,1)} - \tilde{f}(x) = \int_0^\pi \psi(t)\tilde{K}_n(t)dt$$

$$= \left[\int_0^{1/n} + \int_{1/n}^\delta + \int_\delta^\pi\right]\psi(t)\tilde{K}_n(t)dt$$

$$= J_1 + J_2 + J_3 \text{ (say)} \tag{5.5}$$

Let

$$|J_1| = \int_0^{1/n} |\psi(t)||\tilde{K}_n(t)|dt$$

$$= O\left[\int_0^{1/n} \frac{1}{t}|\psi(t)|dt\right] \text{ by Lemma 3}$$

$$= O(n)\left[\int_0^{1/n} |\psi(t)|dt\right]$$

$$= O(n)\left[o\left\{\frac{1}{n\alpha(n)}\right\}\right]$$

$$= o\left\{\frac{1}{\alpha(n)}\right\}$$

$$= o(1), \text{ as } n \to \infty \tag{5.6}$$

$$|J_2| = \int_{1/n}^\delta |\psi(t)||\tilde{K}_n(t)|dt$$

$$= O\left[\int_{1/n}^\delta \frac{1}{t}|\psi(t)|dt\right] \text{ by Lemma 4}$$

$$= O\left[\left\{\frac{1}{t}\Psi(t)\right\}_{1/n}^\delta + \int_{1/n}^\delta \frac{1}{t^2}\Psi(t)dt\right]$$

$$= O\left[o\left\{\frac{1}{\alpha(1/t)}\right\}_{1/n}^\delta + \int_{1/n}^\delta o\left\{\frac{1}{t\alpha(1/t)}\right\}dt\right]$$

$$= O\left[o\left\{ \frac{1}{\alpha(n)} \right\} + \int_{1/\delta}^{n} o\left\{ \frac{1}{u\alpha(u)} \right\} du \right]$$

$$= o\left\{ \frac{1}{\alpha(n)} \right\} + o\left\{ \frac{1}{n\alpha(n)} \right\} \int_{1/\delta}^{n} du$$

Using second-mean value theorem for the integral in the second term as $\alpha(n)$ is monotonic

$$= o(1) + o(1) \text{ , as } n \to \infty$$

$$= o(1) \text{ , as } n \to \infty \tag{5.7}$$

Finally,

$$|J_3| = \int_{\delta}^{\pi} |\psi(t)| \left| \tilde{K}_n(t) \right| dt = o(1) \text{ , as } n \to \infty \tag{5.8}$$

By using Riemann-Lebesgue theorem and regularity condition of summability.

Combining (5.6), (5.7) and (5.8) we have

$$\tilde{t}_n^{(E,1)(E,1)} - \tilde{f}(x) = o(1) \text{ , as } n \to \infty$$

This completes the proof of theorem 2.

6. Conclusion

In the field of summability theory, various results pertaining $(E,1)$, $(E,1)X$ and $X(E,1)$ summabilities of Fourier series as well as its Allied series have been reviewed. In future, the present work can be extended to establish new results under certain conditions.

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