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Statistical Inferences Related to the Threshold Values of Uniform and Exponential Distributions

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Abstract: *The estimation of threshold values by various statistical approaches has been intensively studied since 2002. However, confidence intervals and hypothesis testing procedures for the threshold parameters of uniform and exponential distributions have been neglected in threshold value-related inferential studies. The thresholds of random variables with assumed distributions are not always known, and must sometimes be inferred. In this work, the unknown threshold values of uniform and exponential distributions are inferred from unbiased maximum likelihood estimators and their probability density functions. Procedures for constructing confidence intervals and tests of hypotheses related to the unknown threshold values are established. Simulated data are generated from a uniform and an exponential distribution, both with unknown threshold parameters, using the statistical software MINITAB. Based on the simulated data, confidence intervals are constructed and hypotheses are tested on the unknown threshold values. The simulation results almost perfectly agree with the theoretical analysis. Using the proposed procedures, researchers could estimate the threshold parameters of other distributions in future inference-making studies.*

Keywords: *Uniform and Exponential Distributions, the Distributions of First order Statistics, Unbiased Estimators, Confidence Intervals and Tests of Hypotheses for the threshold values.*

1. INTRODUCTION

Estimating the various parameters of certain distributions is a popular topic in statistics. For example, the two-parameter exponential distribution (in which one parameter is the threshold value) has been applied to the analysis of lifetime data [1], and the inventory management of hazardous items [2]. Erdem [3] studied inferential procedures for the parameters of triangular distributions, and Petropoulos [4] showed that the first-order statistic is sufficient for determining the threshold parameter of the two-parameter exponential distribution. The intervals of this distribution have also been estimated [5]. Bhushal and Sipakar [6] estimated the threshold point of growth and inflation in Nepal. Massacci [7] estimated unknown thresholds using large-dimensional threshold factor models. The impact of missing thresholds was investigated by Riley et al. [8] in a meta-analysis context. Morfeld et al. [9] estimated the threshold values in the distributions of respirable quartz dust concentration and silicosis incidence in a cohort of German porcelain workers. Mann [10] stated that “Earth will cross the climate danger threshold by 2036”. However, to my knowledge, interval estimation and hypotheses testing of the threshold parameters of uniform and exponential distributions has not been reported.

1.1. Uniform and Exponential Distributions with Unknown Threshold Values

Exponential and uniform distributions are commonly encountered in physical situations. For example, the amount of gasoline sold daily at a service station may be uniformly distributed between a minimum and maximum of 2,000 and 5,000 gallons, respectively. Similarly, the waiting time of a passenger at a subway station may uniformly vary between 1 and 15 minutes.

Examples of random variables following exponential distributions are the length of time between telephone calls (or between arrivals at a service station), and the lifetimes of electronic components (i.e., time to internal failure).

In this work, the unknown threshold parameters of uniform and exponential distributions are estimated from unbiased maximum likelihood estimators (MLEs) and their probability density functions. Procedures for constructing the confidence intervals and tests of hypotheses related to these unknown threshold parameters are also established. To date, such procedures have not been reported for the threshold parameters of these distributions.

The threshold parameter of a random variable is not always known. However, as the threshold parameter θ provides an estimate of the earliest time to failure (or success), it is an important parameter of random variables, and may need to be inferred.

To estimate the unknown parameter of interest, a random sample of size *n* is taken from the distribution (population). In general, the MLE provides a plausible estimator. The MLEs of the threshold parameters of uniform and exponential distributions constitute the first-order statistics in the present study.

This work presents unbiased estimators for the threshold parameters of the uniform and exponential distributions, and establishes procedures for constructing confidence intervals and testing hypotheses of these parameters. The theoretical findings are validated in simulation studies of data generated by the statistical software MINITAB. The simulation results are summarized in Section 6, and presented in full (along with the computations) in the Appendices.

1.2. Probability Density Function (Pdf) and Cumulative Distribution Function (Cdf) of the First-Order Statistic

Let $X_1, X_2, ..., X_n$ be a random sample of size *n* from a continuous distribution over the interval (c, d), and let Y_i be the *i*th-order statistic (*i* =1, 2,... *n*).

The formula for computing the pdf of the *i*th-order statistic is found in almost all mathematical statistics textbooks, including Roussas [11]:

$$f_{Y_i}(y) = \frac{n!}{(i-1)!(n-i)!}[F(y)]^{(i-1)}[1-F(y)]^{(n-i)} f_Y(y) \quad , c < y < d \, , \tag{1}$$

where *n* is the size of the random sample, $F(y)$ is the cumulative distribution function (cdf), and $f_{Y_i}(y)$ is the pdf of the *i*th-order statistic Y_i.

When *i* =1, the pdf reduces to:

$$f_{Y_1}(y) = n[1-F(y)]^{(n-1)} f_Y(y), \quad c < y < d \tag{2}$$

The cumulative distribution function (cdf) of the first-order statistic is given by

$$F(y) = P(Y_1 \le y) = \int_c^y n[1-F(y)]^{(n-1)} f_Y(y)dy = [1-F(c)]^{(n)} - [1-F(y)]^{(n)} \tag{3}$$

1.3. Properties of the First-Order Statistic From $X \sim Uniform(\theta, b)$

Assuming real values in the interval (θ, b), a uniformly-distributed continuous random variable has the pdf

$$f(x) = \frac{1}{b-\theta}, \theta < x < b \, , \tag{4}$$

Where *b* is the known upper bound value, and θ is the unknown threshold parameter of the uniformly distributed random variable *X*.

The MLE of the threshold parameter θ is the first-order statistic of a random sample of size *n*, namely, $Y_1 = Min\{X_1, X_2, ..., X_n\}$.

The pdf given by (4) has the following cdf:

$$F(y) = \int_\theta^y \frac{1}{b-\theta} dx = \frac{y-\theta}{b-\theta}, \qquad \theta < y < b \, . \tag{5}$$

Equations (2) give the following pdf for Y_1:

$$f_{Y_1}(y) = n\left[1-\frac{y-\theta}{b-\theta}\right]^{(n-1)}\left(\frac{1}{b-\theta}\right) = \frac{n}{(b-\theta)^n}(b-y)^{(n-1)}, \theta < y < b. \tag{6}$$

The expected value and variance of Y_1 is then computed as follows:

$$E(Y_1^k) = \frac{n}{(b-\theta)^n}\int_\theta^b y^k (b-y)^{(n-1)} dy$$

where k is the order of the moment of $Y_1\,(k = 1,2)$.

$$E(Y_1) = \frac{n\theta+b}{(n+1)},\ E(Y_1^2) = \frac{n(b-\theta)^2}{(n+2)},\ Var(Y_1) = \frac{n(b-\theta)^2}{(n+1)^2(n+2)} \tag{7}$$

From (7) the unbiased estimator of θ is given as

$$T_1 = \frac{(n+1)Y_1 - b}{n} \to E(T_1) = \theta \tag{8}$$

$$Var(T_1) = \left(\frac{n+1}{n}\right)^2 Var(Y_1) = \frac{(b-\theta)^2}{n(n+2)}. \tag{9}$$

2. Confidence Interval for the Threshold Parameter θ of the Distribution $X \sim Uniform(\theta, b)$

From the following probability statements, we construct a $100(1-\alpha)\%$ confidence interval for θ as (where α denotes the level of significance)

$$P(y_{1L} < Y_1 < y_{1U}) = (1-\alpha). \tag{10}$$

Equivalently we can write,

$$F(y_{1L}) = P(Y_1 < y_{1L}) = \frac{\alpha}{2}, and\ F(y_{1U}) = P(Y_1 < y_{1U}) = 1-\frac{\alpha}{2}.$$

From Eq. (5), we obtain

$$1-F(y) = 1-\frac{y-\theta}{b-\theta} = \frac{b-y}{b-\theta}, \quad \theta < y < b.$$

Substituting this expression into (3), we have

$$F(y_{1L}) = P(Y_1 < y_{1L}) = \left[\frac{b-\theta}{b-\theta}\right]^n - \left[\frac{b-y_{1L}}{b-\theta}\right]^n = 1-\left[\frac{b-y_{1L}}{b-\theta}\right]^n = \frac{\alpha}{2},$$

from which y_{1L} is obtained as

$$y_{1L} = b-(b-\theta)\left(1-\frac{\alpha}{2}\right)^{1/n} \tag{11}$$

Similarly,

$$F(y_{1U}) = P(Y_1 < y_{1U}) = \left[\frac{b-\theta}{b-\theta}\right]^n - \left[\frac{b-y_{1U}}{b-\theta}\right]^n = 1-\left[\frac{b-y_{1U}}{b-\theta}\right]^n = 1-\frac{\alpha}{2}$$

$$y_{1U} = b-(b-\theta)\left(\frac{\alpha}{2}\right)^{1/n}. \tag{12}$$

Substituting the results of (11) and (12) into (10), we obtain

$$P\left(b-(b-\theta)\left(1-\frac{\alpha}{2}\right)^{1/n} < Y_1 < b-(b-\theta)\left(\frac{\alpha}{2}\right)^{1/n}\right) = (1-\alpha)\,.$$

Solving the above inequalities for θ, the $(1-\alpha)100\%$ confidence interval (in terms of the first order statistic Y_1) of the threshold parameter θ is obtained as

$$\left(b-\frac{b-Y_1}{\left(\frac{\alpha}{2}\right)^{1/n}},\ b-\frac{b-Y_1}{\left(1-\frac{\alpha}{2}\right)^{1/n}}\right). \tag{13}$$

This confidence interval can also be expressed in terms of the unbiased estimate T_1 of the threshold parameter θ:

$$\left(b-\frac{n(b-T_1)}{(n+1)\left(\frac{\alpha}{2}\right)^{1/n}},\ b-\frac{n(b-T_1)}{(n+1)\left(1-\frac{\alpha}{2}\right)^{1/n}}\right) . \tag{14}$$

3. Tests of Hypotheses Related to the Threshold Value θ of the Random Variable $X \sim Uniform(\theta, b)$

When testing $H_0: \theta = \theta_0$, $H_0: \theta \le \theta_0$ or $H_0: \theta \ge \theta_0$ against the proper alternative hypothesis, a plausible test statistic is $Y_1 = X_{Min}$, for which the unbiased estimator of θ (given by Eq. (1.8)) is a linear function of Y_1.

For the chosen significance level α , the following decision rules apply.

Table 1. *Tests of Hypotheses Related to the Threshold value* θ *for the Distribution* $X \sim Uniform(\theta, b)$

$H_0: \theta = \theta_0$ $H_1: \theta \ne \theta_0$	$H_0: \theta \le \theta_0$ $H_1: \theta > \theta_0$	$H_0: \theta \ge \theta_0$ $H_1: \theta < \theta_0$
If $Y_1 \ge y_{1U}$ or $Y_1 \le y_{1L}$ H_0 is rejected	If $Y_1 \ge y_{1U}$ H_0 is rejected	If $Y_1 \le y_{1L}$ H_0 is rejected
Don't reject H_0 otherwise	Don't reject H_0 otherwise	Don't reject H_0 otherwise
Where, $y_{1L} = b-(b-\theta_0)\left(1-\frac{\alpha}{2}\right)^{1/n}$ and $y_{1U} = b-(b-\theta_0)\left(\frac{\alpha}{2}\right)^{1/n}$	Where, $y_{1U} = b-(b-\theta_0)(\alpha)^{1/n}$	Where $y_{1L} = b-(b-\theta_0)(1-\alpha)^{1/n}$

Note that in the procedures for constructing confidence intervals and testing hypotheses, we include the observed value of the first order statistic Y_1 and the chosen significance level α .

4. Confidence Interval for the Threshold Parameter β of the Exponential Distribution

Let Y be an exponentially distributed continuous random variable with an unknown threshold parameter β and a known parameter λ. The pdf of Y is given by

$$f_Y(y;\beta,\lambda) = e^{\beta/\lambda}\frac{1}{\lambda}e^{-y/\lambda}\,;\; y > \beta > 0,\; \lambda > 0 \tag{15}$$

$$\begin{aligned} &F(y) = P(Y \le y) = e^{\beta/\lambda}\int_{\beta}^{y}\frac{1}{\lambda}e^{-y/\lambda}dt = 1 - e^{\beta/\lambda}e^{-y/\lambda},\; y > \beta > 0 \rightarrow \\ &1 - F(y) = 1 - P(Y \le y) = e^{\beta/\lambda}e^{-y/\lambda},\; y > \beta > 0 \end{aligned} \quad . \tag{16}$$

The MLE of β is the first-order statistic of a random sample of size *n*.

Let W_1 be the first-order statistic of a random sample of size *n* from this distribution. The pdf of W_1 is given by

$$f_{W_1}(w,\beta,\lambda) = e^{n\beta/\lambda}\frac{n}{\lambda}e^{-n\,w/\lambda},\; w > \beta > 0\,,$$

$$E(W_1) = \int_{\beta}^{\infty} w e^{n\beta/\lambda}\frac{n}{\lambda}e^{-nw/\lambda}dw = \beta + \frac{\lambda}{n}, \tag{17}$$

$$E(W_1^2) = \int_{\beta}^{\infty} w^2 e^{n\beta/\lambda}\frac{n}{\lambda}e^{-nw/\lambda}dw = \left(\beta + \frac{\lambda}{n}\right)^2 + \frac{\lambda^2}{n^2}$$

$$Var(W_1) = \frac{\lambda^2}{n^2} \quad . \tag{18}$$

From (17) we can derive an unbiased estimator for the unknown threshold parameter β .

$$T_2 = W_1 - \frac{\lambda}{n}, \quad Var(T_2) = Var(W_1 - \frac{\lambda}{n}) = Var(W_1) = \frac{\lambda^2}{n^2} \tag{19}$$

From the probability expression $P(w_L < W_1 < w_U) = 1-\alpha$, we can determine a $100(1-\alpha)\%$ confidence interval for β . The above probability expression may also be expressed as

$$P(W_1 < w_L) = \frac{\alpha}{2} \;\; and \;\; P(W_1 > w_U) = \frac{\alpha}{2}.$$

Using Eqs. (3) and (16), we obtain

$$P(W_1 < w_L) = -\left[e^{\beta/\lambda}e^{-y/\lambda}\right]^n \Big|_{\beta}^{w_L} = 1 - \left[e^{\beta/\lambda}e^{-w_L/\lambda}\right]^n = \frac{\alpha}{2}\;, \tag{20}$$

from which w_L is obtained as

$$w_L = \beta - \frac{\lambda}{n}\ln\left(1 - \frac{\alpha}{2}\right). \tag{21}$$

Similarly,

$$P(W_1 < w_U) = -\left[e^{\beta/\lambda}e^{-y/\lambda}\right]^n \Big|_{\beta}^{w_U} = 1 - \left[e^{\beta/\lambda}e^{-w_U/\lambda}\right]^n = 1 - \frac{\alpha}{2} \quad . \tag{22}$$

From (22), we obtain

$$w_U = \beta - \frac{\lambda}{n}\ln\left(\frac{\alpha}{2}\right). \tag{23}$$

Inserting (21) and (23) in to $P(w_L < W_1 < w_U) = 1-\alpha$, we obtain the intended confidence interval for the threshold parameter β; namely,

$$P[w_1+\frac{\lambda}{n}\ln(\alpha/2)<\beta<w_1+\frac{\lambda}{n}\ln(1-\alpha/2)]=(1-\alpha) \tag{24}$$

This confidence interval may also be expressed in terms of the unbiased estimate T_2 of the threshold β

$T_2=\left(w_1-\frac{\lambda}{n}\right)\rightarrow w_1=\left(T_2+\frac{\lambda}{n}\right)$; that is,

$$P(T_2+\frac{\lambda}{n}(1+\ln(\alpha/2))<\beta<T_2+\frac{\lambda}{n}(1+\ln(1-\alpha/2)))=(1-\alpha). \tag{25}$$

5. Hypotheses Tests Related to the Threshold Parameter β of the Exponential Distribution

When testing $H_0:\beta=\beta_0$, $H_0:\beta\le\beta_0$ or $H_0:\beta\ge\beta_0$ against to the proper alternative hypothesis, a plausible test statistic is $W_1=X_{Min}$.

For the chosen significance level α , the decision rules listed in Table 2 are applicable.

Table 2. *Tests of Hypotheses Related to the Threshold Value β of the Distribution* $W\sim Exponential(\lambda;\beta)$

$H_0:\beta=\beta_0$ $H_1:\beta\ne\beta_0$	$H_0:\beta\le\beta_0$ $H_1:\beta>\beta_0$	$H_0:\beta\ge\beta_0$ $H_1:\beta<\beta_0$
If $W_1\ge w_U$ or $W_1\le w_L$, reject H_0	If $W_1\ge w_U$, reject H_0	If $W_1\le w_L$, reject H_0
Otherwise, do not reject H_0	Otherwise, do not reject H_0	Otherwise, do not reject H_0
where $w_L=\beta_0-\frac{\lambda}{n}\ln\left(1-\frac{\alpha}{2}\right)$ and $w_U=\beta_0-\frac{\lambda}{n}\ln\left(\frac{\alpha}{2}\right)$	where $w_U=\beta_0-\frac{\lambda}{n}\ln(\alpha)$	where $w_L=\beta_0-\frac{\lambda}{n}\ln(1-\alpha)$

6. Simulation Studies

To validate the theoretical findings of the studied distributions, I simulated a uniform distribution with threshold parameter θ over the interval $(\theta=5, b=10)$, and on an exponential distribution with threshold parameter β = 4 and $\lambda=0.2$.

It is a known fact that, for any distribution, the cumulative distribution function $F(y)$ is a random variable with a uniform distribution over the interval (0, 1).

Using MINITAB [12] software, 100 independent random samples of sizes 100 were drawn from a uniform distribution over the interval (0, 1). ***These observations were considered as the sampled $F(y)$ values.***

To obtain corresponding random observations from a uniform distribution over the interval (θ, b) , I applied an inverse transformation to the cdf. The $[F(y)]$ of this distribution becomes

$$F(y)=\frac{y-\theta}{b-\theta}\rightarrow y=\theta+(b-\theta)F(y). \tag{26}$$

The sampled $F(y)$ values were used to simulate the uniform distribution over the interval $(\theta=5, b=10)$. Equation (26) yields $y=5+(10-5)F(y)=5+5F(y)$ values. From each random sample of size 100, the first-order statistic y_1 was determined, and unbiased estimator values were computed. The simulation results are summarized in Table 3. As indicated in the table, $E(T_1) \cong \theta = 5$ with a very small coefficient of variation.

[Coefficient of Variation for $T_1 = 0.00289/5.00116 \approx 5.78 \times 10^{-4}$].

Table 3. *Simulation Summaries for Uniform Distribution*

	Mean	Variance
Y_1	***5.05065***	***0.01806***
T_1	***5.00116***	***0.00289***
Lower Confidence Limit (LCL)	***4.69968***	
Upper Confidence Limit (UCL)	***5.0494***	

From each random sample of size 100, the 95% confidence intervals of θ were computed by the rule derived from (14). Ninety-seven of these 100 confidence intervals contained the true value (= 5) of the threshold parameter θ.

Table 4. *Simulation summaries for the hypotheses tests Related to the threshold parameter of the random variable* $X \sim Uniform(\theta = 5, b = 10)$

$H_0 : \theta = 5$ $H_1 \theta \neq 5$ If $Y_1 \geq y_{1U}$ or $Y_1 \leq y_{1L}$ H_0 is rejected	$y_{1L} = 10 - (10-5)(0.975)^{1/100} \cong 5.001266$ $y_{1U} = 10 - (10-5)(0.025)^{1/100} \cong 5.18108$
In 6 out of 100 tests, $H_0 : \theta = 5$ was rejected at the $\alpha = 0.05$ significance level.	

For ease of visualization, the proportions of accepted confidence intervals and hypotheses after the simulations are graphically presented in Fig. 1. The charts summarize the results of the uniform distribution.

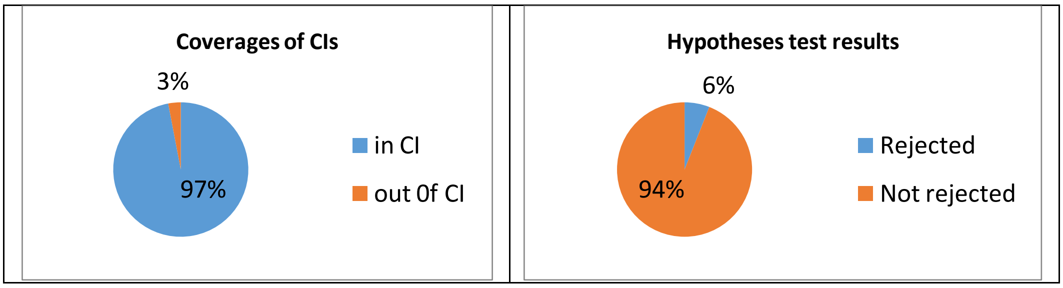

Fig. 1. *Pie charts showing the proportions of accepted and rejected confidence intervals (CI) and hypotheses in the simulated uniform distribution.*

The hypotheses in Table 4 were tested on the same simulated raw data described at the beginning of this section. The decisions were classified into rejecting or not rejecting the null hypotheses. At the chosen level of significance (α = 0.05), the true null hypothesis was rejected in only 6 out of 100 tests.

Similar procedures were followed for the threshold parameter β of the exponential distribution. One hundred samples of size 100 were generated from the uniform distribution over the interval (0, 1). These were converted into 100 samples from an exponential distribution with the probability density function $f_Y(y; \beta, \lambda) = e^{\beta/\lambda} \frac{1}{\lambda} e^{-y/\lambda}$, $y > \beta > 0$, $\lambda > 0$ by an inverse transformation applied on the following cumulative probability distribution function $F(y) = 1 - e^{\beta/\lambda} e^{-y/\lambda}$, $y > \beta > 0$. The transformation procedure is described below.

Let $F(y_i) = u_i$ be a randomly generated observation from a uniform distribution over the interval (0, 1). To convert this to a random observation from the exponential distribution, we set $u_i = 1 - e^{\beta/\lambda} e^{-y_i/\lambda}$, yielding

$$y_i = \beta - \lambda \ln(1 - u_i). \tag{27}$$

Inserting λ = 0.2 and β = 4 into (27), the required observations are generated through $y_i = 4 - 0.2\ln(1-u_i)$. From each sample of size 100, we determine the smallest observed (simulated) values (the w_i values).

The simulation results are summarized in Table 5. According to this table, $E(T_2) \cong \beta = 4$ with a very small coefficient of variation.

[Coefficient of Variation for T_2 = 0.00000304/3.999968 ≈ 7.60 × 10^{-7}].

The 95% confidence intervals of β were computed by (4.10). Ninety-seven out of 100 confidence intervals contained the chosen threshold (β = 4).

Table 5. *Simulation summaries for exponential distribution*

	Mean	Variance
W_1	***4.001901***	***0.00000304***
T_2	***3.999968***	***0.00000304***
Lower Confidence Limit (LCL)	***3.994591***	
Upper Confidence Limit (UCL)	***4.001918***	

Table 6. *Simulation summaries for the hypotheses tests related to the threshold value of the random variable* $W \sim Exponential(\lambda = 0{,}2;\ \beta = 4)$

$H_0 : \beta = 4$ $H_1 : \beta \neq 4$ If $W_1 \geq w_U$ or $W_1 \leq w_L$, reject H_0	where $w_L = 4 - \dfrac{0.2}{100}\ln(0.975) \equiv 4.000050636$ $w_U = 4 - \dfrac{0.2}{100}\ln(0.025) \cong 4.007377759$
$H_0 : \beta = 4$ was rejected in 3 out of 100 tests	

The confidence intervals and hypothesis test results after the simulations are graphically presented in Fig. 2.

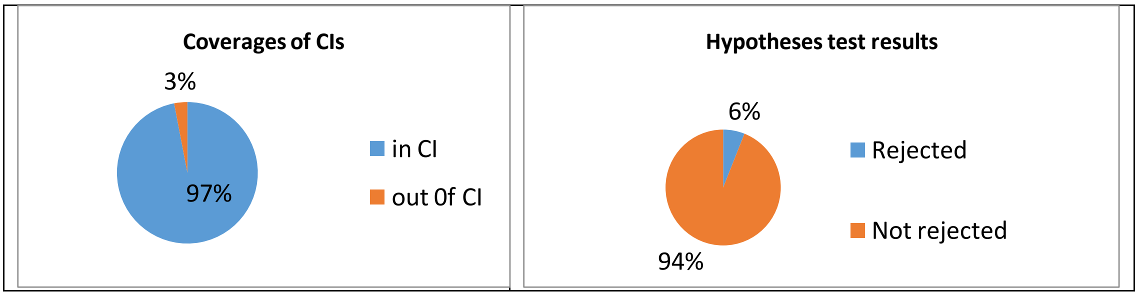

Fig. 2. *Pie charts showing the proportions of accepted and rejected confidence intervals (CI) and hypotheses in the simulated exponential distribution.*

The hypotheses in Table 6 were tested on the same simulated raw data described at the beginning of this section. Again, the decisions are classified into rejecting or not rejecting the null hypotheses. At the chosen significance level (α = 0.05), the true null hypothesis was rejected in only 3 out of 100 tests.

The observed values from the uniform distributions were simulated in the MINITAB [12] package. The detailed computations are tabulated in Appendices I and II.

7. Conclusion

This study obtained the unbiased MLEs in uniform and exponential distributions. Procedures for determining the confidence intervals and hypotheses tests were established from the pdfs and cdfs of these estimators. The developed procedures might also infer the threshold parameters of other distributions. Although this study investigated the unknown lower bound threshold of the random variable, the method is equally applicable to the upper bound of the random variable of interest.

Inferencing of threshold parameters will significantly contribute to decision making problems in real life situations. For instance, when issuing a warranty statement on a machine component, one needs to know the shortest lifetime of the component. If this is unknown but the component lifetime follows a certain distribution (such as uniform or exponential), the above procedures may assist the decision maker in estimating the threshold value, and determining the pdf and cdf of the random variable of interest. Thus, the developed procedures provide robust and reliable information for composing the warranty statement.

As another example, when deciding whether an athlete qualifies for an Olympic 100-m dash race, the committee needs to determine the upper bound of the qualifying completion time. For this purpose, they may estimate the upper confidence limit of the threshold parameter from the 100-m dash completion times of the world's top competitors in recent past events (such as the last 5 Olympics).

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Ismail Erdem, Ph.D. from North Carolina State University (USA). Has quite lengthy teaching/administrative experience. Worked for North Carolina State and North Carolina Central Universities for 5 years. Later, worked for Middle East Technical University (METU) Ankara/Turkey for 16 years. After METU, joined Baskent University Ankara-Turkey, has been teaching statistics and quantitative methods for 39 years. Main interest areas are: Mathematical and Applied Statistics, Time Series Analysis, and Forecasting. He has more than 40 National/International publications in science journals and conference materials.

APPENDIX I

	Simulated Data from Uniform Distribution $(\theta_1 = 5, b = 10)$						
Sample	n	Y_1	T_1	LCL	UCL	Is $H_0: \theta = 5$ Rejected?	Does CI Contain $\theta = 5$?
C1	100	5,036	4,98636	4,68399	5,03474		
C2	100	5,009	4,95909	4,65508	5,00774		
C3	100	5,095	5,04595	4,74718	5,09376		
C4	100	5,008	4,95808	4,65401	5,00674		
C5	100	5,028	4,97828	4,67542	5,02674		
C6	100	5,025	4,97525	4,67221	5,02374		
C7	100	5,025	4,97525	4,67221	5,02374		
C8	100	5,027	4,97727	4,67435	5,02574		
C9	100	5,005	4,95505	4,65079	5,00374		
C10	100	5,059	5,00959	4,70862	5,05775		
C11	100	5,252	5,20452	4,91531	5,2508	Yes	
C12	100	5,07	5,0207	4,7204	5,06875		
C13	100	5,006	4,95606	4,65186	5,00474		
C14	100	5,052	5,00252	4,70113	5,05075		
C15	100	5,007	4,95707	4,65294	5,00574		
C16	100	5,026	4,97626	4,67328	5,02474		
C17	100	5,073	5,02373	4,72362	5,07175		
C18	100	5,076	5,02676	4,72683	5,07475		
C19	100	5,12	5,0712	4,77395	5,11876		
C20	100	5,107	5,05807	4,76003	5,10576		
C21	100	5,139	5,09039	4,7943	5,13777		
C22	100	5,008	4,95808	4,65401	5,00674		
C23	100	5,022	4,97222	4,669	5,02074		
C24	100	5,103	5,05403	4,75574	5,10176		
C25	100	5,27	5,2227	4,93458	5,2688	Yes	
C26	100	5,036	4,98636	4,68399	5,03474		
C27	100	5,105	5,05605	4,75788	5,10376		
C28	100	5,179	5,13079	4,83713	5,17778		
C29	100	5,039	4,98939	4,6872	5,03774		
C30	100	5,01	4,9601	4,65615	5,00874		
C31	100	5,049	4,99949	4,69791	5,04775		
C32	100	5,004	4,95404	4,64972	5,00273		
C33	100	5,025	4,97525	4,67221	5,02374		
C34	100	5,041	4,99141	4,68935	5,03974		
C35	100	5,037	4,98737	4,68506	5,03574		
C36	100	5,033	4,98333	4,68078	5,03174		
C37	100	5,094	5,04494	4,7461	5,09276		
C38	100	5,054	5,00454	4,70327	5,05275		
C39	100	5,029	4,97929	4,6765	5,02774		

C40	100	5	4,95	4,64544	4,99873	Yes	No
C41	100	5,019	4,96919	4,66579	5,01774		
C42	100	5,003	4,95303	4,64865	5,00173		
C43	100	5,033	4,98333	4,68078	5,03174		
C44	100	5,1	5,051	4,75253	5,09876		
C45	100	5,062	5,01262	4,71184	5,06075		
C46	100	5,024	4,97424	4,67114	5,02274		
C47	100	5,058	5,00858	4,70755	5,05675		
C48	100	5,06	5,0106	4,70969	5,05875		
C49	100	5,062	5,01262	4,71184	5,06075		
C50	100	5,043	4,99343	4,69149	5,04174		
C51	100	5,036	4,98636	4,68399	5,03474		
C52	100	5,038	4,98838	4,68613	5,03674		
C53	100	5,011	4,96111	4,65722	5,00974		
C54	100	5,014	4,96414	4,66043	5,01274		
C55	100	5,016	4,96616	4,66257	5,01474		
C56	100	5,016	4,96616	4,66257	5,01474		
C57	100	5,056	5,00656	4,70541	5,05475		
C58	100	5,005	4,95505	4,65079	5,00374		
C59	100	5,065	5,01565	4,71505	5,06375		
C60	100	5,069	5,01969	4,71933	5,06775		
C61	100	5,01	4,9601	4,65615	5,00874		
C62	100	5,001	4,95101	4,64651	4,99973	Yes	No
C63	100	5,007	4,95707	4,65294	5,00574		
C64	100	5,066	5,01666	4,71612	5,06475		
C65	100	5,009	4,95909	4,65508	5,00774		
C66	100	5,022	4,97222	4,669	5,02074		
C67	100	5,131	5,08231	4,78573	5,12977		
C68	100	5,024	4,97424	4,67114	5,02274		
C69	100	5,01	4,9601	4,65615	5,00874		
C70	100	5,004	4,95404	4,64972	5,00273		
C71	100	5,001	4,95101	4,64651	4,99973	Yes	No
C72	100	5,123	5,07423	4,77716	5,12177		
C73	100	5,019	4,96919	4,66579	5,01774		
C74	100	5,128	5,07928	4,78252	5,12677		
C75	100	5,044	4,99444	4,69256	5,04275		
C76	100	5,014	4,96414	4,66043	5,01274		
C77	100	5,17	5,1217	4,82749	5,16878		
C78	100	5,031	4,98131	4,67864	5,02974		
C79	100	5,012	4,96212	4,65829	5,01074		
C80	100	5,009	4,95909	4,65508	5,00774		
C81	100	5,036	4,98636	4,68399	5,03474		
C82	100	5,025	4,97525	4,67221	5,02374		
C83	100	5,109	5,06009	4,76217	5,10776		

C84	100	5,032	4,98232	4,67971	5,03074		
C85	100	5,156	5,10756	4,8125	5,15477		
C86	100	5,012	4,96212	4,65829	5,01074		
C87	100	5,062	5,01262	4,71184	5,06075		
C88	100	5,008	4,95808	4,65401	5,00674		
C89	100	5,004	4,95404	4,64972	5,00273		
C90	100	5,038	4,98838	4,68613	5,03674		
C91	100	5,204	5,15604	4,8639	5,20279	Yes	
C92	100	5,015	4,96515	4,6615	5,01374		
C93	100	5,114	5,06514	4,76752	5,11276		
C94	100	5,01	4,9601	4,65615	5,00874		
C95	100	5,006	4,95606	4,65186	5,00474		
C96	100	5,028	4,97828	4,67542	5,02674		
C97	100	5,074	5,02474	4,72469	5,07275		
C98	100	5,041	4,99141	4,68935	5,03974		
C99	100	5,055	5,00555	4,70434	5,05375		
C100	100	5,028	4,97828	4,67542	5,02674		
	Mean	5,05065	5,00116	4,69968	5,0494		
	Variance	0,00284	0,00289	0,00325	0,00284		
	Min	5	4,95	4,64544	4,99873		
	Max.	5,27	5,2227	4,93458	5,2688		

The null hypothesis $H_0 : \theta = 5$ is rejected in 6 out of 100 tests.

Three of the 95% confidence intervals exclude the threshold $\theta_1 = 5$.

APPENDIX II

	Simulation Data from Exponential Distribution ($\beta = 4, \lambda = 0{,}2$)					
Sample No	n	W_1	LCL	UCL	Is $H_0 : \beta = 5$ Rejected?	Does CI Contain $\beta = 5$?
C1	100	4,003104	3,995726	4,003053		
C2	100	4,002658	3,99528	4,002607		
C3	100	4,00201	3,994632	4,001959		
C4	100	4,001647	3,994269	4,001596		
C5	100	4,00022	3,992842	4,000169		
C6	100	4,004347	3,996969	4,004296		
C7	100	4,007956	4,000578	4,007906	Yes	No
C8	100	4,001647	3,994269	4,001596		
C9	100	4,001526	3,994148	4,001475		
C10	100	4,00002	3,992642	3,999969	Yes	No
C11	100	4,00036	3,992983	4,00031		
C12	100	4,001083	3,993705	4,001032		
C13	100	4,001687	3,994309	4,001636		
C14	100	4,000561	3,993183	4,00051		
C15	100	4,00024	3,992862	4,00019		
C16	100	4,001143	3,993766	4,001093		
C17	100	4,003328	3,99595	4,003277		

C18	100	4,002597	3,995219	4,002546		
C19	100	4,001627	3,994249	4,001576		
C20	100	4,002273	3,994895	4,002222		
C21	100	4,001345	3,993967	4,001294		
C22	100	4,002232	3,994855	4,002182		
C23	100	4,004245	3,996867	4,004194		
C24	100	4,0003	3,992922	4,00025		
C25	100	4,002172	3,994794	4,002121		
C26	100	4,004204	3,996826	4,004153		
C27	100	4,001546	3,994168	4,001495		
C28	100	4,002435	3,995057	4,002384		
C29	100	4,002698	3,99532	4,002647		
C30	100	4,004061	3,996683	4,00401		
C31	100	4,000882	3,993504	4,000831		
C32	100	4,000601	3,993223	4,00055		
C33	100	4,005577	3,998199	4,005526		
C34	100	4,00016	3,992782	4,000109		
C35	100	4,00014	3,992762	4,000089		
C36	100	4,001043	3,993665	4,000992		
C37	100	4,000621	3,993243	4,00057		
C38	100	4,003145	3,995767	4,003094		
C39	100	4,001043	3,993665	4,000992		
C40	100	4,002496	3,995118	4,002445		
C41	100	4,00286	3,995483	4,00281		
C42	100	4,005968	3,99859	4,005918		
C43	100	4,00004	3,992662	3,999989	Yes	No
C44	100	4,00028	3,992902	4,00023		
C45	100	4,000942	3,993564	4,000892		
C46	100	4,004654	3,997276	4,004603		
C47	100	4,000481	3,993103	4,00043		
C48	100	4,00018	3,992802	4,000129		
C49	100	4,003633	3,996255	4,003582		
C50	100	4,001224	3,993846	4,001173		
C51	100	4,000461	3,993083	4,00041		
C52	100	4,00347	3,996092	4,003419		
C53	100	4,005413	3,998035	4,005362		
C54	100	4,002071	3,994693	4,00202		
C55	100	4,001083	3,993705	4,001032		
C56	100	4,000641	3,993263	4,00059		
C57	100	4,008248	4,00087	4,008197		
C58	100	4,001748	3,99437	4,001697		
C59	100	4,005556	3,998179	4,005506		
C60	100	4,001365	3,993987	4,001314		
C61	100	4,004347	3,996969	4,004296		
C62	100	4,00032	3,992942	4,00027		
C63	100	4,001445	3,994067	4,001395		
C64	100	4,00034	3,992963	4,00029		

C65	100	4,001304	3,993926	4,001254		
C66	100	4,001003	3,993625	4,000952		
C67	100	4,00006	3,992682	4,000009		
C68	100	4,001808	3,99443	4,001758		
C69	100	4,001445	3,994067	4,001395		
C70	100	4,001345	3,993967	4,001294		
C71	100	4,0003	3,992922	4,00025		
C72	100	4,000922	3,993544	4,000871		
C73	100	4,001526	3,994148	4,001475		
C74	100	4,00203	3,994653	4,00198		
C75	100	4,000982	3,993605	4,000932		
C76	100	4,00018	3,992802	4,000129		
C77	100	4,004695	3,997317	4,004644		
C78	100	4,001768	3,99439	4,001717		
C79	100	4,000481	3,993103	4,00043		
C80	100	4,000902	3,993524	4,000851		
C81	100	4,002293	3,994915	4,002242		
C82	100	4,000842	3,993464	4,000791		
C83	100	4,000581	3,993203	4,00053		
C84	100	4,00014	3,992762	4,000089		
C85	100	4,003206	3,995828	4,003155		
C86	100	4,00014	3,992762	4,000089		
C87	100	4,002516	3,995138	4,002465		
C88	100	4,006051	3,998673	4,006		
C89	100	4,001606	3,994229	4,001556		
C90	100	4,003328	3,99595	4,003277		
C91	100	4,004183	3,996806	4,004133		
C92	100	4,002658	3,99528	4,002607		
C93	100	4,002253	3,994875	4,002202		
C94	100	4,000922	3,993544	4,000871		
C95	100	4,000601	3,993223	4,00055		
C96	100	4,000761	3,993384	4,000711		
C97	100	4,00042	3,993043	4,00037		
C98	100	4,002759	3,995381	4,002708		
C99	100	4,000902	3,993524	4,000851		
C100	100	4,002192	3,994814	4,002141		
	Mean	4,001968	3,994591	4,001918		
	Min	4,00002	3,992642	3,999969		
	Max	4,008248	4,00087	4,008197		
	Var	3,04E-06	3,04E-06	3,04E-06		

The null hypothesis $H_0 : \beta = 4$ is rejected in 3 out of 100 tests.

Three of the 95% confidence intervals exclude the threshold $\beta = 4$.