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A Generalized Finite Hankel Type Transformation and a Parseval Type Equation

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Abstract: *In this paper, we study the finite Hankel type transformation on spaces of generalized functions by developing new procedure. Two Hankel type integral transformations $h_{\alpha,\beta}$ and $h^*_{\alpha,\beta}$ are considered and they satisfy Parseval type equation defined by (1.2). We have defined a space $S_{\alpha,\beta}$ of functions and a space $L_{\alpha,\beta}$ of complex sequences and it is further shown that $h^*_{\alpha,\beta}$ and $h'_{\alpha,\beta}$ are isomorphisms from $S_{\alpha,\beta}$ onto $L_{\alpha,\beta}$ and $S'_{\alpha,\beta}$ onto $L'_{\alpha,\beta}$ respectively. Finally some applications of new generalized finite Hankel type transformation are established*

Keywords: *Finite Hankel type transformation, Parseval type equation, generalized finite Hankel type transformation.*

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1. INTRODUCTION

I.N. Sneddon [14] was first to introduce finite Hankel transforms of classical functions. The same was later studied by [3], [4], [7], [15]. Recently Zemanian [18], Pandey and Pathak [11] and Pathak [12] extended these transforms to certain spaces of distributions as a special case of general theory on orthonromal series expansions of generalized functions. Dube [5], Pathak and Singh [13] and Mendez and Negrin [10] investigated finite Hankel transformations in other spaces of distributions through a procedure quite different from that one which was developed in [18] and [12].

We define finite Hankel type transformation of the first kind by

$$\left(h_{\alpha,\beta}f\right)(n) = \int_0^1 x\ J_{\alpha-\beta}\,(\lambda_n\,x)\,f(x)\,dx,\ \ n = 0,1,2,\dots\dots$$

for $(\alpha-\beta) \geq -\frac{1}{2}$, where J_ν denotes the Bessel function of the first kind and order ν and λ_n, $n = 0,1,2,\dots\dots$, represent the positive roots of $J_{\alpha-\beta}(x) = 0$ arranged in ascending order of magnitude [17, p.596].

For $(\alpha-\beta) \geq -\frac{1}{2}$ and $a \geq \frac{1}{2}$, we introduce the space $U_{\alpha,\beta,a}$ of finitely differentiable functions on (0,1) such that

$$\rho_k^{\alpha,\beta,a}(\phi) = Sup_{0<x<1}\left|x^{a-1}\,B^{*k}_{\alpha,\beta}\,\phi(x)\right| < \infty, \text{for every } k \in \mathbb{N},$$

where $B^*_{\alpha,\beta} = x^{-(\alpha-\beta)}D\,x^{4\alpha}\,D\,x^{-(3\alpha+\beta)}$.

$U_{\alpha,\beta,a}$ is equipped with the topology generated by the family of seminorms $\left\{\rho_k^{\alpha,\beta,a}\right\}_{k=0}^{\infty}$. Thus $U_{\alpha,\beta,a}$ is a Frechet space. $U'_{\alpha,\beta,a}$ denotes the dual of $U_{\alpha,\beta,a}$ and is endowed with the weak topology.

For $f \in U'_{\alpha,\beta,a}$, the generalized finite Hankel type transform of f is defined by

$$F(n) = \langle f(x)\,, x\,J_{\alpha-\beta}\,(\lambda_n x)\rangle,\ \ for\ \ n = 0,1,2,\dots\dots \tag{1.1}$$

Our main objective in this paper is to define the finite Hankel type transformation $h_{\alpha,\beta}$ on new spaces of generalized functions by developing a new procedure.

We introduce the finite Hankel type transformation $h^*_{\alpha,\beta}$ by

$$\left(h^*_{\alpha,\beta} f\right)(n) = \frac{2}{J^2_{3\alpha+\beta}(\lambda_n)} \int_0^1 J_{\alpha-\beta}(\lambda_n x)\, f(x)\, dx,\ \ n = 0,1,2,\ldots\ldots.$$

when $(\alpha-\beta) \geq -\frac{1}{2}$.

The transformation $h_{\alpha,\beta}$ and $h^*_{\alpha,\beta}$ are closely connected and they satisfy the Parseval equation

$$\sum_{n=0}^{\infty}\left(h_{\alpha,\beta} f\right)(n)\left(h^*_{\alpha,\beta}\phi\right)(n) = \int_0^1 f(x)\,\phi(x)\,dx \qquad (1.2)$$

when $(\alpha-\beta) \geq -\frac{1}{2}$ and f and ϕ are suitable functions.

We define a space $S_{\alpha,\beta}$ of functions and a space $L_{\alpha,\beta}$ of sequences and we prove that $h^*_{\alpha,\beta}$ is an isomorphism from $S_{\alpha,\beta}$ onto $L_{\alpha,\beta}$ provided that $(\alpha-\beta) \geq -\frac{1}{2}$. The generalized finite Hankel type transformation $h_{\alpha,\beta}\, f$ of $f \in S'_{\alpha,\beta}$, the dual of $S_{\alpha,\beta}$, is defined through

$$<\left(h'_{\alpha,\beta} f\right), \left(\left(h^*_{\alpha,\beta}\,\phi\right)(n)\right)_{n=0}^{\infty}> = <f,\phi>,\ for\ \phi \in S_{\alpha,\beta}. \qquad (1.3)$$

One can notice that (1.3) appears as a generalization of the Parseval equation (1.2).

We show that the conventional finite Hankel type transformation $h_{\alpha,\beta}$ and generalized finite Hankel type transformation given by (1.1) are special cases of our generalized transformation.

Throughout this paper $(\alpha-\beta)$ denotes a real number greater or equal to $-\frac{1}{2}$.

Now require some properties of Bessel functions.

The behaviours of $J_{\alpha-\beta}$ near the origin and the infinity are the following ones:

$$J_{\alpha-\beta}(x) = O\left(x^{\alpha-\beta}\right),\ as\ x \to 0^+, \qquad (1.4)$$

$$J_{\alpha-\beta}(x) \backsimeq \left(\frac{2}{\pi x}\right)^{\alpha+\beta}\left[Cos\left(x-\frac{1}{2}(\alpha-\beta)\pi-\frac{\pi}{4}\sum_{m=0}^{\infty}\frac{(-1)^m(\alpha-\beta,2m)}{(2x)^{2m}}\right) - Sin\left(x-\frac{1}{2}(\alpha-\beta)\pi-\frac{1}{4}\pi\sum_{m=0}^{\infty}\frac{(-1)^m(\alpha-\beta,2m+1)}{(2x)^{2m+1}}\right)\right],$$

as $x \to \infty$, (1.5)

where $(\alpha-\beta,k)$ is understood as in Watson [17, p.198]

The main differentiation formulae for $J_{\alpha-\beta}$ are

$$\frac{d}{dx}\left(x^{\alpha-\beta} J_{\alpha-\beta}(xy)\right) = y\, x^{\alpha-\beta} J_{-\alpha-3\beta}(xy), \qquad (1.6)$$

$$\frac{d}{dx}\left(x^{-(\alpha-\beta)} J_{\alpha-\beta}(xy) = -y x^{-(\alpha-\beta)} J_{3\alpha+\beta}(xy)\right), \qquad (1.7)$$

for $x,y > 0$. By combining (1.6) and (1.7), it can be easily inferred

$$B_{\alpha,\beta}\, J_{\alpha-\beta}(x) = -J_{\alpha-\beta}(x),\ for\ x > 0, \qquad (1.8)$$

where $B_{\alpha,\beta} = x^{-(3\alpha+\beta)} D\, x^{4\alpha}\, D x^{-(\alpha-\beta)}$.

2. The Spaces $S_{\alpha,\beta}$ AND $L_{\alpha,\beta}$ and the Finite Hankel Type Transformation

Definition 2.1: We define the $S_{\alpha,\beta}$ as the space of all complex valued functions $\phi(x)$ on $(0,1]$ such that $\phi(x)$ is infinitely differentiable and satisfies for every $k \in \mathbb{N}$

(i) $B^{**}_{\alpha,\beta}\ \phi\ (1) = 0,$

(ii) $x^{3\alpha+\beta}B^{*k}_{\alpha,\beta}\ \phi(x)\ \rightarrow 0\ and\ x^{4\alpha}\ D\left(x^{-(3\alpha+\beta)}B^{*k}_{\alpha,\beta}\ \phi(x)\right)\ \rightarrow 0, as\ x\ \rightarrow\ 0^{+},$ and

(iii) $x^{-(\alpha+\beta)}B^{*k}_{\alpha,\beta}\ \phi(x)\ \in L\ (0,1)$.

$S_{\alpha,\beta}$ is endowed with the topology generated by the family of seminorms $\{\|\phi\|_k\}_{k=0}^{\infty}$,

where

$$\|\phi\|_k = \int_0^1 x^{-(\alpha+\beta)}\left|B^{*k}_{\alpha,\beta}\ \phi\ (x)\right| dx,\ \ for\ \phi\ \in\ S_{\alpha,\beta}\ and\ k\ \in\ \mathbb{N}.$$

Notice that $\|\phi\|_k$ is a norm for $k = 0$. $S_{\alpha,\beta}$ is a Hausdorff topological linear space that verifies the first countability axiom. Moreover, the operator $B^{*}_{\alpha,\beta}$ defines a continuous mapping from $S_{\alpha,\beta}$ into itself. $S'_{\alpha,\beta}$ is the dual space of $S_{\alpha,\beta}$ and it is equipped with the usual weak topology.

We require the following result which will be useful in the sequel.

Lemma 2.2: If $f(x)$ is a function defined on (0,1) such that $x^{\alpha+\beta}f(x)$ is bounded on (0,1), then $f(x)$ generates a member of $S'_{\alpha,\beta}$ through the definition

$$< f(x), \phi\ (x) > = \int_0^1 f(x)\ \phi(x)\ dx,\ \ \phi\ \in\ S_{\alpha,\beta}\ .$$

Proof: The result easily follows from the following inequality

$$|< f(x),\ \phi\ (x) >|\ \leq\ \|\phi\|_0\ Sup_{0<x<1}\left|x^{\alpha+\beta}\ f(x)\right|,\ \ \phi\ \in\ S_{\alpha,\beta}\ .$$

Lemma 2.3: Let $(\alpha - \beta)\ \geq\ -1/2$ and $a\ \geq\ {}^{1}/{}_{2}$. Then $S_{\alpha,\beta}\ \subset\ U_{\alpha,\beta,a}$ and the topology of $S_{\alpha,\beta}$ is stronger than that induced on it by $U_{\alpha,\beta,a}$.

Proof: Let $\phi\ \in\ S_{\alpha,\beta}$. In view of the conditions (i) and

(ii) of Definition 2.1, we can write

$$x^{a-1}\ B^{*k}_{\alpha,\beta}\ \phi\ (x) =\ x^{a+\alpha-\beta}\int_1^x t^{-4\alpha}\int_0^t u^{\alpha-\beta}\ B^{*k+1}_{\alpha,\beta}\phi(u)\ du\ dt$$

for every $x\ \in\ (0,1)\ and\ k\ \in\ \mathbb{N}.$

Therefore

$$\left|x^{a-1}\ B^{*k}_{\alpha,\beta}\ \phi\ (x)\right|\ \leq\ x^{a+\alpha-\beta}\int_x^1 t^{2\beta-1}\ dt\int_0^1 u^{-(\alpha+\beta)}\left|B^{*k+1}_{\alpha,\beta}\ \phi\ (u)\right| d\mu$$

$$\leq\ x^{a-(\alpha+\beta)}\int_0^1 u^{-(\alpha+\beta)}\ \left|B^{*k+1}_{\alpha,\beta}\ \phi(u)\right| du$$

$$\leq\ \int_0^1 u^{-(\alpha+\beta)}\ \left|B^{*k+1}_{\alpha,\beta}\ \phi\ (u)\right| du\ \ ,\ \ for\ every\ \ x\ \in\ (0,1)\ and\ k\ \in\ \mathbb{N}.$$

Hence, for every $\phi\ \in\ S_{\alpha,\beta}$ and $k\ \in\ \mathbb{N},$

$$\underset{0<x<1}{Sup}\left|x^{a-1}\ B^{*k}_{\alpha,\beta}\ \phi(x)\right|\ \leq\ \ \|\phi\|_{k+1}\ ,$$

and $S_{\alpha,\beta}$ is contained in $U_{\alpha,\beta,a}$ and the inclusion is continuous. Thus proof is completed.

Remark: From Lemma 2.3, we can deduce that if $f\ \in\ U'_{\alpha,\beta,a}$, then the restriction of f to $S_{\alpha,\beta}$ is a member of $S'_{\alpha,\beta}$.

Definition 2.4: We define $L_{\alpha,\beta}$ as the space of all complex sequences $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} a_n\,\lambda_n^{2k} = 0$, for every $k\in\mathbb{N}$, where $\lambda_{n,}\ n = 0,1,2,\ldots.$ represent the positive roots of the equation $J_{\alpha-\beta}(x) = 0$ arranged in ascending order of magnitude.

The topology of $L_{\alpha,\beta}$ is that generated by the family of norms $\{\gamma_{\alpha,\beta}^{k}\}_{k=0}^{\infty}$, where

$$\gamma_{\alpha,\beta}^{k}((a_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |a_n|\,\lambda_n^{2k}\,,\ \ for\ ((a_n)_{n=0}^{\infty}) \in L_{\alpha,\beta}\ and\ k \in \mathbb{N}.$$

Notice that $\gamma_{\alpha,\beta}^{k}\ ((a_n)_{n=0}^{\infty}) < \infty$ for every $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}$.

Thus $L_{\alpha,\beta}$ is Hausdorff topological linear space that satisfies the first countability axiom. $L'_{\alpha,\beta}$ denotes the dual space of $L_{\alpha,\beta}$ and it is endowed with the weak topology.

Now we introduce continuous operations in $L_{\alpha,\beta}$ and $L'_{\alpha,\beta}$ in the following Lemma.

Lemma 2.5: Let $(b_n)_{n=0}^{\infty}$ be a complex sequence such that $|b_n| \le M\,\lambda_n^{l}$ for every $n\in\mathbb{N}$ and for some $l\in\mathbb{N}$ and $m > 0$.

Then the linear operator

$$(a_n)_{n=0}^{\infty} \to (a_n b_n)_{n=0}^{\infty}$$

is a continuous mapping from $L_{\alpha,\beta}$ into itself.

Moreover the operator in $L'_{\alpha,\beta}$, $B \to (b_n)_{n=0}^{\infty}\,B,$ where

$$\langle (b_n)_{n=0}^{\infty}\,B,\ (a_n)_{n=0}^{\infty}\rangle = \langle B, \quad (a_n b_n)_{n=0}^{\infty}\rangle, \qquad for\ (a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}\,,$$

is a continuous mapping from $L'_{\alpha,\beta}$ into itself.

Proof: It is enough to see that

$$\gamma_{\alpha,\beta}^{k}((a_n b_n)_{n=0}^{\infty}) \le M \sum_{n=0}^{\infty} |a_n|\,\lambda_n^{2k+l} \le M_1\,\gamma_{\alpha,\beta}^{k+l}\ ((a_n)_{n=0}^{\infty}),$$

for $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}$ and $k\in\mathbb{N}$, where M_1 being a suitable positive constant. This completes the proof.

Lemma 2.6: If $(b_n)_{n=0}^{\infty}$ is a complex sequence satisfying the same conditions as in Lemma 2.5, then $(b_n)_{n=0}^{\infty}$ generates a member of $L'_{\alpha,\beta}$ by

$$\langle (b_n)_{n=0}^{\infty}\,,\ (a_n)_{n=0}^{\infty}\rangle = \textstyle\sum_{n=0}^{\infty} a_n b_n\ ,\ for\ (a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}.$$

Now we state our main theorem of this section.

Theorem 2.7: For $(\alpha-\beta) \ge -\frac{1}{2}$, the finite Hankel type transformation $h_{\alpha,\beta}^{*}$ is an isomorphism from $S_{\alpha,\beta}$ onto $L_{\alpha,\beta}$.

Proof: Let $\phi \in S_{\alpha,\beta}$. As it is known, $h_{\alpha,\beta}^{*}\,\phi = (a_n)_{n=0}^{\infty}$, where

$a_n = \frac{2}{J_{3\alpha+\beta}^{2}(\lambda_n)} \int_0^1 J_{\alpha-\beta}\,(\lambda_n x)\,\phi\,(x)\,dx,$ for every $n\in\mathbb{N}$.

In view of the operational rule (1.6), we can write for every $n\in\mathbb{N}$

$$\lambda_n^2\,a_n = \frac{2\lambda_n^2}{J_{3\alpha+\beta}^{2}\,(\lambda_n)} \int_0^1 J_{\alpha-\beta}\,(\lambda_n x)\,\phi\,(x)\,dx$$

$$= \frac{2\lambda_n}{J_{3\alpha+\beta}^{2}\,(\lambda_n)} \int_0^1 \frac{d}{dx}\left(x^{3\alpha+\beta}\,J_{3\alpha+\beta}\,(\lambda_n x)\right)\,x^{-(3\alpha+\beta)}\phi(x)\,dx$$

$$= \frac{2\,\lambda_n}{J^2_{3\alpha+\beta}(\lambda_n)} \left[\left(J_{3\alpha+\beta}\,(\lambda_n x)\,\phi(x) \right)_0^1 - \int_0^1 x^{3\alpha+\beta}\, J_{3\alpha+\beta}\,(\lambda_n x)\,\frac{d}{dx}\left(x^{-(3\alpha+\beta)}\phi(x) \right)\,dx \right].$$

However, by (1.4), $\left(J_{3\alpha+\beta}(\lambda_n x)\phi\,(x) \right]_0^1 = 0,$ since $\phi(1) = 0$ and

$$\lim_{x\to 0^+} x^{3\alpha+\beta}\,\phi(x) = 0.$$

Hence

$$\lambda_n^2 a_n = \frac{2\lambda_n}{J^2_{3\alpha+\beta}\,(\lambda_n)} \int_0^1 x^{3\alpha+\beta}\,J_{3\alpha+\beta}(\lambda_n x)\,\frac{d}{dx}\left(x^{-(3\alpha+\beta)}\phi(x) \right)\,dx. \tag{2.1}$$

Now, by invoking (1.7), we have

$$\lambda_n \int_0^1 x^{3\alpha+\beta}\,J_{3\alpha+\beta}\,(\lambda_n x)\,\frac{d}{dx}\left(x^{-(3\alpha+\beta)}\phi(x) \right)\,dx$$

$$= -\int_0^1 \frac{d}{dx}\left(x^{-(\alpha-\beta)}\,J_{\alpha-\beta}(\lambda_n x) \right)\,x^{4x}\,\frac{d}{dx}\left(x^{-(3\alpha+\beta)}\phi(x) \right)\,dx$$

$$= \left[-J_{\alpha-\beta}\,(\lambda_n x)\,x^{3\alpha+\beta}\,\frac{d}{dx}\left(x^{-(3\alpha+\beta)}\phi(x) \right) \right]_0^1 + \int_0^1 B^*_{\alpha,\beta}\,\phi(x)\,J_{\alpha-\beta}\,\,(\lambda_n x)\,dx.$$

The limit terms are equal to zero by (1.4) because $J_{\alpha-\beta}\,(\lambda_n) = 0,\ \phi \in C^\infty\left((0,1] \right)$,

$$\lim_{x\to 0^+} x^{4\alpha}\,\frac{d}{dx}\left(x^{-(3\alpha+\beta)}\phi(x) \right) = 0.$$

Therefore

$$\lambda_n \int_0^1 x^{3\alpha+\beta} J_{3\alpha+\beta}(\lambda_n\,x)\,\frac{d}{dx}\,\left(x^{-(3\alpha+\beta)}\phi(x) \right)dx = \int_0^1 B^*_{\alpha,\beta}\,\,\phi(x)\,J_{\alpha,\beta}\,(\lambda_n\,x)\,dx \tag{2.2}$$

Using relations (2.1) and (2.2), we obtain

$$a_n\,\lambda_n^2 = -\frac{2}{J^2_{3\alpha+\beta}\,(\lambda_n)} \int_0^1 B^*_{\alpha,\beta}\,\phi(x)\,J_{\alpha-\beta}\,(\lambda_n\,x)\,dx, \quad \text{for every } n \in \mathbb{N}.$$

By induction, we have

$$\lambda_n^{2k} a_n = (-1)^k\,\frac{2}{J^2_{3\alpha+\beta}\,(\lambda_n)} \int_0^1 B^{*k}_{\alpha,\beta}\,\phi\,(x)\,J_{\alpha-\beta}\,(\lambda_n x)\,dx, \quad \text{for every } n,\ k \in \mathbb{N}. \tag{2.3}$$

From (2.3), according to Riemann-Lebesgue Lemma ([17, p. 457]), we have

$$J^2_{3\alpha+\beta}\,(\lambda_n)\,\lambda_n^{2k}\,a_n \to 0,\ as\,n \to \infty.$$

Moreover by (1.5), there exists a positive constant M such that

$$\lambda_n^{2k}\,|a_n| \le M\,J^2_{3\alpha+\beta}(\lambda_n)\,\lambda_n^{2k+1}\,|a_n|\,,$$

and then $\lambda_n^{2k}\,a_n \to 0,\ as\,n \to \infty,\ for\ every\ k \in \mathbb{N}.$

Now, for certain M_i , $i = 0,1,2,$

$$\sum_{n=0}^{\infty} \lambda_n^{2k}\,|a_n| = \sum_{n=0}^{\infty} \frac{2}{J^2_{3\alpha+\beta}(\lambda_n)\,\lambda_n^4} \left| \int_0^1 B^{*k+2}_{\alpha,\beta}\,\phi(x)\,J_{\alpha-\beta}\,(\lambda_n x)\,dx \right|$$

$$\le M_1 \sum_{n=0}^{\infty} \lambda_n^{-5(\alpha+\beta)} \int_0^1 \left| (\lambda_n x)^{\alpha+\beta} J_{\alpha-\beta}\,(\lambda_n x) \right| x^{-(\alpha+\beta)}\,\left| B^{*k+2}_{\alpha,\beta}\,\phi\,(x) \right| dx$$

$$\le M_2 \sum_{n=0}^{\infty} \lambda_n^{-2} \int_0^1 x^{-(\alpha+\beta)}\,\left| B^{*k+2}_{\alpha,\beta}\,\phi\,(x) \right| dx.$$

Since

$$\sum_{n=0}^{\infty} \lambda_n^{-2} < \infty,$$

we get

$\gamma_{\alpha,\beta}^{k}\left((a_n)_{n=0}^{\infty}\right) \leq M_3 \|\phi\|_{k+2}$, $for\ every\ k \in \mathbb{N}$ and $\phi \in S_{\alpha,\beta}$ and for some $M_3 > 0$.

This inequality proves that the linear mapping $h_{\alpha,\beta}^{*}$ is continuous from $S_{\alpha,\beta}$ into $L_{\alpha,\beta}$.

Now let $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}$ and define

$$\tau_{\alpha,\beta}\left((a_n)_{n=0}^{\infty}\right)(x) = \phi(x) = \sum_{n=0}^{\infty} a_n x J_{\alpha-\beta}(\lambda_n x), \ for\ x \in (0,1].$$

By (1.4) and (1.5), we have

$$\sum_{n=0}^{\infty} \left| a_n x J_{\alpha-\beta}(\lambda_n x) \right| \leq M\, x^{\alpha+\beta} \sum_{n=0}^{\infty} |a_n|, \ x > 0$$

for a suitable $M > 0$. Thus $\phi(x) \in C(0,\infty)$. In a similar way we can prove that $\phi \in C^{\infty}(0,\infty)$. Moreover by invoking (1.8) we obtain

$$B_{\alpha,\beta}^{*k}\,\phi(x) = \sum_{n=0}^{\infty} (-1)^k a_n \lambda_n^{2k}\, x J_{\alpha-\beta}(\lambda_n x), \ for\ x > 0$$

and $k \in \mathbb{N}$.

Then $B_{\alpha,\beta}^{*k}\,\phi(1) = 0$, for each $k \in \mathbb{N}$.

We can also infer

$$\left| x^{3\alpha+\beta} B_{\alpha,\beta}^{*k}\,\phi(x) \right| \leq M_1\, x^{2(2\alpha+\beta)} \sum_{n=0}^{\infty} |a_n|\, \lambda_n^{2k}, \ for\ x > 0\ and\ k \in \mathbb{N},$$

and from (1.4), (1.5) and (1.6) ,

$$\left| x^{4\alpha} \frac{d}{dx}\left(x^{-(3\alpha+\beta)} B_{\alpha,\beta}^{*k}\,\phi(x) \right) \right| \leq M_2\, x^{2(3\alpha+\beta)} \sum_{n=0}^{\infty} |a_n|\, \lambda_n^{2k+5\alpha+3\beta},$$

for $x > 0$ and $k \in \mathbb{N}$.

Here $M_1\ and\ M_2$ are suitable positive constants. Hence

$$\lim_{x\to 0^+} x^{3\alpha+\beta} B_{\alpha,\beta}^{*k}\,\phi(x) = \lim_{x\to 0^+} x^{4\alpha} \frac{d}{dx}\left(x^{-(3\alpha+\beta)} B_{\alpha,\beta}^{*k}\,\phi(x) \right) = 0,$$

$for\ every\ k \in \mathbb{N}$.

On the other hand, as the series defining $B_{\alpha,\beta}^{*k}\,\phi(x)$ is uniformly convergent in $x \in (0,1)$, there exists a positive constant M_3 such that

$$\int_0^1 x^{-(\alpha+\beta)} \left| B_{\alpha,\beta}^{*k}\,\phi(x) \right| dx \leq M_3 \sum_{n=0}^{\infty} |a_n|\, \lambda_n^{2k},$$

for every $k \in \mathbb{N}$.

Therefore $\tau_{\alpha,\beta}$ is a continuous mapping from $L_{\alpha,\beta}$ into $S_{\alpha,\beta}$. Finally from Watson [17,p.59], we can infer that

$$\left(\tau_{\alpha,\beta}.\ h_{\alpha,\beta}^{*}\right)\phi = \phi,\ for\ \phi \in S_{\alpha,\beta},\ and$$

$$\left(h_{\alpha,\beta}^{*}.\tau_{\alpha,\beta}\right)(a_m)_{n=0}^{\infty} = (a_m)_{n=0}^{\infty},\ for\ (a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}.$$

This completes the proof.

3. THE GENERALIZED FINITE HANKEL TYPE TRANSFORMATION

We define the generalized finite Hankel type transformation $h'_{\alpha,\beta}$ on $S'_{\alpha,\beta}$ as follows.

$$\langle\left(h'_{\alpha,\beta}\, f\right), \left(h^{*}_{\alpha,\beta}\, \phi\right)\, (n)_{n=0}^{\infty}\rangle = \langle f(x),\ \phi(x)\rangle \tag{3.1}$$

for every $\phi \in S_{\alpha,\beta}$. Notice that (3.1) appears as a generalization of the Parseval equation (1.2).

From Zemanian [19, Theorem 1.10-2] and Theorem 2.7, we immediately obtain

Theorem 3.1: For $(\alpha-\beta) \geq -1/2$, the generalized finite Hankel type transformation $h'_{\alpha,\beta}$ is an isomorphism from $S'_{\alpha,\beta}$ onto $L'_{\alpha,\beta}$.

In the following theorem, we establish that the conventional finite Hankel type transformation $h_{\alpha,\beta}$ is a special case of the generalized finite Hankel type transformation defined in (3.1).

Theorem 3.2: Let $f(x)$ be a function defined on (0,1) such that $x^{\alpha+\beta} f(x)$ is bounded on (0,1). Then $\left(\left(h_{\alpha,\beta} f\right)(n)\right)_{n=0}^{\infty}$ agrees with $\left(h'_{\alpha,\beta} f\right)$ as members of $L'_{\alpha,\beta}$.

Proof: The conventional finite Hankel type transformation of f is defined by

$$\left(h_{\alpha,\beta} f\right)(n) = \int_0^{\prime} x\, J_{\alpha-\beta}\,(\lambda_n x)\, f(x)\, dx,\ \ for\ n \in \mathbb{N}.$$

Then as $x^{\alpha+\beta} f(x)$ is bounded on (0,1) and by (1.4) and (1.5) we can write

$$\begin{aligned}\left|\left(h_{\alpha,\beta}\, f\right)(n)\right| &\leq M\, \lambda_n^{-(\alpha+\beta)} \int_0^1 \left|(\lambda_n x)^{\alpha+\beta} J_{\alpha-\beta}\,(\lambda_n x)\right| dx \\ &\leq M_1\, \lambda_n^{-(\alpha+\beta)}, \qquad for\ n \in \mathbb{N},\end{aligned}$$

where M and M_1 are certain positive constants.

Therefore in view of Lemma 2.6, $\left(\left(h_{\alpha,\beta}\, f\right)(n)\right)_{n=0}^{\infty}$ generates a member of $L'_{\alpha,\beta}$ by

$$\begin{aligned}\langle\left(\left(h_{\alpha,\beta} f\right)(n)\right)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\rangle &= \sum_{n=0}^{\infty}\left(h_{\alpha,\beta} f\right)(n)\, a_n \\ &= \sum_{n=0}^{\infty} a_n \int_0^1 x\, J_{\alpha-\beta}\,(\lambda_n x)\, f(x)\, dx \\ &= \int_0^1 f(x) \sum_{n=0}^{\infty} a_n\, x\, J_{\alpha-\beta}\,(\lambda_n x)\, dx,\end{aligned}$$

for every $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}$.

The last equality is justified since the series

$$\sum_{n=0}^{\infty} a_n\, x^{\alpha+\beta}\, J_{\alpha-\beta}\,(\lambda_n x)$$

is uniformly convergent on (0,1) and $x^{\alpha+\beta} f(x)$ is bounded on (0,1).

We can also write

$$\langle\left(\left(h_{\alpha,\beta} f\right)(n)\right)_{n=0}^{\infty},\ \left(\left(h_{\alpha,\beta}\phi\right)(n)\right)_{n=0}^{\infty}\rangle$$

$$= \int_0^1 f(x) \sum_{n=0}^{\infty} \left(h^*_{\alpha,\beta}\phi\right)(n)\, x\, J_{\alpha-\beta}(\lambda_n x)\, dx = \int_0^1 f(x)\, \phi(x)\, dx$$

for every $\phi \in S_{\alpha,\beta}$.

Hence according Lemma 2.2, we conclude

$\langle \left(\left(h_{\alpha,\beta}f\right)(n)\right)_{n=0}^{\infty}, \left(\left(h^*_{\alpha,\beta}\phi\right)(n)\right)_{n=0}^{\infty} \rangle = < f(x), \phi(x),\ for\ \phi \in S_{\alpha,\beta}$ and $\left(\left(h_{\alpha,\beta}f\right)(n)\right)_{n=0}^{\infty} = \left(h'_{\alpha,\beta} f\right)$ as members of $L'_{\alpha,\beta}$. Thus proof is completed.

We now prove that the generalized finite Hankel type transform of f given by (1.1) is equal (in the sense of equality in $L'_{\alpha,\beta}$) to the generalized finite Hankel type transform of f as given by (3.1).

Theorem 3.3: Let $(\alpha-\beta) \geq -1/2\ ,\ a \geq 1/2\ and\ f \in U'_{\alpha,\beta,a}$.

Then

$\langle \left(F(n)\right)_{n=0}^{\infty},\ (a_n)_{n=0}^{\infty} \rangle = \langle \left(h'_{\alpha,\beta}f\right),\ (a_n)_{n=0}^{\infty} \rangle$, for every $(a_n)_{n=0}^{\infty} \in L'_{\alpha,\beta}$,

where,

$$F(n) = \langle f(x), \quad x\, J_{\alpha-\beta}(\lambda_n x) \rangle, \qquad for\ every\ n \in \mathbb{N}.$$

Proof: By Zemanian [19, Theorem 1.8-1], since $\in U'_{\alpha,\beta,a}$, there exist $r \in \mathbb{N}$ and $M > 0$ such that

$$\left|\langle f(x), \quad x\, J_{\alpha-\beta}(\lambda_n x)\rangle\right| \leq M \max_{0\leq k\leq r} \operatorname{Sup}_{0<x<1} \left|x^{a-1} B^{*k}_{\alpha,\beta} \left(x\, J_{\alpha-\beta}(\lambda_n x)\right)\right|,$$

for every $n \in \mathbb{N}$.

Hence, by (1.4), (1.5) and (1.8), we can infer that

$$|F(n)| \leq M\ max_{0\leq k\leq r}\, Sup_{0<x<1} \left|x^{a-1} \lambda_n^{2k} x\, J_{\alpha-\beta}(\lambda_n x)\right| \leq M_1\, \lambda_n^{2r} \tag{3.2}$$

for a certain $M_1 > 0$. By invoking Lemma 2.6, (3.2) proves that $\left(F(n)\right)_{n=0}^{\infty}$ generates a member of $L'_{\alpha,\beta}$ through

$$\langle \left(F(n)\right)_{n=0}^{\infty}, \quad (a_n)_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} F(n)\, a_n\ ,\ for\ (a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}\ .$$

To prove our assertion, we must establish that

$$\textstyle\sum_{n=0}^{\infty} F(n)\, a_n = \langle f(x),\ \sum_{n=0}^{\infty} a_n x\, J_{\alpha-\beta}(\lambda_n x)\rangle,\ for\ (a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}. \tag{3.3}$$

Let $(a_n)_{n=0}^{\infty} \in L_{\alpha,\beta}$. As it is easy to see,

$$\textstyle\sum_{n=0}^{\infty} F(n)\, a_n = \langle f(x), \sum_{n=0}^{m} a_n x\, J_{\alpha-\beta}(\lambda_n x)\rangle + \sum_{n=m+1}^{\infty} a_n \langle f(x), x\, J_{\alpha-\beta}(\lambda_n x)\rangle. \tag{3.4}$$

for every $m \in \mathbb{N}$.

By using (3.2), we can obtain

$$\left|\sum_{n=m+1}^{\infty} a_n \langle f(x), \quad x\, J_{\alpha-\beta}(\lambda_n x)\rangle\right| \leq M_1 \sum_{n=m+1}^{\infty} |a_n|\, \lambda_n^{2r}\ ,$$

for every $m \in \mathbb{N}$ with $M_1 > 0$. Then

$$\textstyle\lim_{m\to\infty} \sum_{n=m+1}^{\infty} a_n \langle f(x),\ x\, J_{\alpha-\beta}(\lambda_n x)\rangle = 0. \tag{3.5}$$

Moreover, for every $k \in \mathbb{N}$ and $x \in (0,1)$, we obtain

$$\left|x^{a-1} B^{*k}_{\alpha,\beta} \left[\sum_{n=m+1}^{\infty} a_n\ x\, J_{\alpha-\beta}(\lambda_n x)\right]\right|$$

$$\leq\ x^{a-1} \sum_{n=m+1}^{\infty} \left|a_n\, x\, J_{\alpha-\beta}(\lambda_n x)\right| \lambda_n^{2k} \leq\ M_2\, x^{a-(\alpha+\beta)} \sum_{n=m+1}^{\infty} |a_n| \lambda_n^{2k}$$

for a suitable $M_2 > 0$.

Hence

$$\underset{0<x<1}{\text{Sup}} \left| x^{a-1}\, B_{\alpha,\beta}^{*k} \left[\sum_{n=m+1}^{\infty} a_n\, x\, J_{\alpha-\beta}\, (\lambda_n x) \right] \right| \leq\ M_2 \sum_{n=m+1}^{\infty} |a_n|\, \lambda_n^{2k}\ ,$$

for every $k\ \in\ \mathbb{N}$, and

$$\sum_{n=m+1}^{\infty} a_n\, x\, J_{\alpha-\beta}\, (\lambda_n x)\ \to 0\, , as\, m\ \to\ \infty, \qquad in\, S_{\alpha,\beta}\ ,$$

because $(a_n)_{n=0}^{\infty}\ \in\ L_{\alpha,\beta}$.

Therefore, since $f\ \in\ S'_{\alpha,\beta}$,

$\lim_{m\to\infty}\langle f(x),\ \sum_{n=m+1}^{\infty} a_n\, x\, J_{\alpha-\beta}\, (\lambda_n x)\rangle = 0$. (3.6)

Now from (3.3), we can conclude

$$\langle \left(F(n)\right)_{n=0}^{\infty}\, ,\ \left(\left(h_{\alpha,\beta}^{*}\phi\right)(n)\right)_{n=0}^{\infty} \rangle = \langle f(x), \sum_{n=0}^{\infty} \left(h_{\alpha,\beta}^{*}\, \phi\right)\ (m)\, x\, J_{\alpha-\beta}\, (\lambda_n x) \rangle$$

$$= \langle f(x), \phi\, (x) \rangle = \langle \left(h'_{\alpha,\beta} f\right), \left(\left(h_{\alpha,\beta}^{*}\phi\right)(n)\right)_{n=0}^{\infty}, \qquad for\, \phi\ \in\ S_{\alpha,\beta}, \rangle$$

and the proof is complete.

4. Application

Firstly we prove an operational-transform formula for the generalized finite Hankel type transformation that will be useful in applications.

Lemma 4.1: Let P be a polynomial and f be in $S'_{\alpha,\beta}$. Then

$$\left(h'_{\alpha,\beta}\, P\, \left(B_{\alpha,\beta}\right) f\right) = P\, (-\lambda_n^2)\, \left(h'_{\alpha,\beta} f\right).$$

Proof: If $f\ \in\ S'_{\alpha,\beta}$, we have

$$\langle \left(h'_{\alpha,\beta}\, P\, \left(B_{\alpha,\beta}\right) f\right), \qquad \left(\left(h_{\alpha,\beta}^{*}\phi\right)(n)\right)_{n=0}^{\infty} \rangle$$

$$= \langle P\, \left(B_{\alpha,\beta}\right) f, \phi \rangle\ =\ \langle f,\ P\, \left(B_{\alpha,\beta}^{*}\right) \phi \rangle$$

$$=\ \langle \left(h'_{\alpha,\beta} f\right),\ \left(\left(h_{\alpha,\beta}^{*}\, P\left(B_{\alpha,\beta}^{*}\right)\phi\right)(n)\right)_{n=0}^{\infty} \rangle$$

$$= \langle \left(h'_{\alpha,\beta} f\right),\ \left(P(-\lambda_n^2)\left(h_{\alpha,\beta}^{*}\, \phi\right)(n)\right)_{n=0}^{\infty} \rangle,$$

$$= \langle P(-\lambda_n^2)\, \left(h'_{\alpha,\beta} f\right),\ \left(\left(h_{\alpha,\beta}^{*}\, \phi\right)(n)\right)_{n=0}^{\infty} \rangle,\ for\ every\ \phi\ \in\ S_{\alpha,\beta}.$$

We consider the functional equation

$$P\left(B_{\alpha,\beta}\right) f = g, \tag{4.1}$$

where g is a given member of $S'_{\alpha,\beta}$, P is a polynomial such that $P\ (-\lambda_n^2)\ \neq 0$ for every $n\ \in\ \mathbb{N}$, and f is unknown generalized function but required to be in $S'_{\alpha,\beta}$.

By applying the generalized finite Hankel type transform to (4.1) and according to Lemma 4.1, we can prove that (4.1) is equivalent to

$$P\ (-\lambda_n^2)\, \left(h'_{\alpha,\beta} f\right)\ =\ \left(h_{\alpha,\beta}^{*}\ g\right).$$

Hence it is not difficult to see that the functional f defined by

$$\langle f,\phi\rangle = \langle g,\sum_{n=0}^{\infty}\frac{1}{P(-\lambda_n^2)}\left(h_{\alpha,\beta}^{*}\phi\right)(n)x\,J_{\alpha-\beta}\,(\lambda_n x)\,\rangle, \qquad for\ \phi\in S_{\alpha,\beta}\,,$$

is in $S'_{\alpha,\beta}$ and it is the solution for (4.1). This completes the proof.

5. Conclusion

This paper provides the study of the finite Hankel type transformation on spaces of generalized functions. The integral transformations $h_{\alpha,\beta}$ and $h^{*}_{\alpha,\beta}$ satisfy Parseval type equation defined above in this paper. We have shown that $h^{*}_{\alpha,\beta}$ and $h'_{\alpha,\beta}$ are isomorphisms from $S_{\alpha,\beta}$ onto $L_{\alpha,\beta}$ and $S'_{\alpha,\beta}$ onto $L'_{\alpha,\beta}$ respectively. Applications of new generalized finite Hankel type transformation established in this paper may be useful in engineering.

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References

[1] Betancor J.J, The Hankel-Schwartz transform for functions of compact support, Rend. Mat. Appl. 7(3-4) (1987), 399-409.

[2] Betancor J.J., A mixed Parseval equation and a generalized Hankel transformation of distributions, Can. J. Math. XLI (2) (1989), 274-284.

[3] Churchill R.V., Fourier Series and Boundary Value Problems, Mc Graw-Hill, New York, 1953.

[4] Cinelli, A extension of the finite Hankel transform and applications, Inst. J. Engng. 3 (1965), 539-559.

[5] Dube L.S., On finite Hankel-transformation of generalized functions, Pacific, J. Math. 62 (1976), 365-378.

[6] Gelfand I.M., Shilov G.E., Generalized functions, Vol. III. Academic Press, New York, 1967.

[7] Liu S.H., Method of generalized finite Hankel transform, Z. Angew. Math. Mech. 51 (1971), 311-313.

[8] Mendez J.M., A mixed Parseval equation and the generalized Hankel transformation, Proc. Amer. Math. Soc. 102 (1988), 619-624.

[9] Mendez J.M., The finite Hankel-Schwartz transform, J. Korean Math. Soc 26(1) (1989), 647-655.

[10] Mendez J.M. and Negrin J.R., Fourier Bessel Series expansion of generalized functions and finite Hankel transforms of distributions, Rev. Roum. de. Math. Pures et. Appl. XXXIV (7) (1989), 647-655.

[11] Pandey J.N., Pathak R.S., Eigen function expansion of generalized functions, Nagoya Math. J.72 (1978), 1-25.

[12] Pathak R.S., Orthogonal series representations for generalized functions, J. Math. Anal. Appl. 130 (1988), 316-333.

[13] Pathak R.S, Singh O.P, Finite Hankel transforms of distributions, Pacific J. Math. 99 (1982), 439-458.

[14] Sneddon I.N., On finite Hankel transforms, Phil. Mag. (7) 17 (1946), 16-25.

[15] Sneddon I.N., The Use of Integral Transforms, Tata Mc Graw. Hill, New Delhi, 1979.

[16] Titchmarsh E.C., A class of expansions in series of Bessel functions, Proc. London Math. Soc. (2) 22 (1924), Xiii-Xvi.

[17] Watson G.N., Theory of Bessel Functions, 2nd.ed., Cambridge University Press, Cambridge, 1958.

[18] Zemanian A. H., Orthonormal series expansions of certain distributions and distributional transform calculus, J. Math. Anal. Appl. 14(1966), 263-275.

[19] Zemanian A. H., Generalized Integral Transformations, Interscience Publishers, New York, 1968.

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