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On the Numerical Ranges

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Abstract: *Properties of operators in a C*-algebra $B(H)$ have been studied by many researchers in operator theory. This paper is an investigation of the numerical ranges of operators in $B(H)$. We show that zero is in the algebraic numerical range of an operator in $B(H)$ if and only if that operator is orthogonal to the identity operator. We then show that the algebraic numerical range of an operator in $B(H)$ is convex and is also equal to the closure of the spatial numerical range of that operator. We employ the inner products of vectors in a Hilbert space H as well as the properties of the states in $B(H)$ in obtaining our results.*

Keywords: *Numerical range, State, Finite operator.*

1. INTRODUCTION

This Let H be a complex Hilbert space with inner product $\langle , \rangle$ and T a bounded linear operator on H. The *spatial numerical range* of $T \in B(H)$ is the set

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

The *maximal numerical range* of $T \in B(H)$ is the set

$$W_o(T) = \{\lambda : \langle Tx_n, x_n\rangle \to \lambda, \|x_n\| = 1, \|Tx_n\| \to \|T\|\}.$$

An element T of a C*-algebra $B(H)$ is said to be *positive* if it is Hermitian and if $\sigma(T) \in \mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive real numbers. We write $T \geq 0$ to mean that T is positive. The set of all positive elements of $B(H)$ is denoted by $B(H)^+$.

A function f on $B(H)$ is a *linear form* if it maps elements of $B(H)$ into $\mathbf{C}$. The set of all linear forms on $B(H)$ is denoted by $B(H)'$. The linear form f is said to be positive on $B(H)$ if $f(T^*T) \geq 0$, for all $T \in B(H)$.

A *state*, f, on the C*-algebra $B(H)$, with unity 1, is a positive linear functional of unit norm. Therefore, if f is a state on $B(H)$, then it is in the dual of $B(H)$, and $f(1) = 1 = \|f\|$. The set of all states on $B(H)$ is denoted by

$$S(B(H)) = \{f \in B(H)' : f(1) = 1 = \|f\|\}.$$

If $T \in B(H)$, then the *algebraic numerical range* of T is the set defined by

$$V(T) = \{f(T) : f \in S(B(H))\}.$$

Let H be a separable infinite dimensional complex Hilbert space. Then a *finite operator* is an operator $T \in B(H)$ for which

$$0 \in W_o(TS - ST) = \overline{W(TS - ST)},$$

for all $S \in B(H)$. The class of all finite operators is denoted by F. A *generalized finite operator* is defined by

$$F(H) = \{(S,T) \in B(H) \times B(H) : \|1 - (SW - WS)\| \geq 1\},$$

for all $W \in B(H)$.

If $S, T \in B(H)$, then S is *orthogonal* to T, written $S \perp T$, if and only if $\|\lambda T\| \leq \|S - \lambda T\|$ for all $\lambda \in \mathbf{C}$.

Williams [4], and Mecheri [3], have worked on finite operators. Williams also considered the orthogonality of an operator in a Banach algebra. Agure [1], [2] has worked on the numerical ranges and norm of derivations. It has been stated in [4] and [3] that the class of finite operators is uniformly closed, contains every normal operator, every operator with the compact direct summand, and the entire C*-algebra generated by each of its members.

In this paper, some results on orthogonality and the numerical range of an element of an algebra are highlighted. It is also proved that the algebraic numerical range $V(T)$ of an operator $T \in B(H)$, is convex, compact, and that it is equal to the closure of the spatial numerical range $W(T)$.

A proof of theorem 4 in [4] is also presented.

As usual, H will represent a complex Hilbert space, and $B(H)$ the algebra of all bounded linear operators on H.

2. Orthogonality of Operator

Williams [4] has proved that 0 is in the algebraic numerical range of $T \in B(H)$ if and only if T is orthogonal to the identity operator, $I \in B(H)$. Below is the proof of this theorem. Recall that if $S, T \in B(H)$, then S is orthogonal to T, written $S \perp T$, if and only if $\|\lambda T\| \leq \|S - \lambda T\|$ for all $\lambda \in \mathbf{C}$.

Theorem 2.1.

Let $T \in B(H)$. Then $0 \in V(T)$ if and only if $|\alpha| \leq \|T - \alpha\|$ for all $\alpha \in \mathbf{C}$ where $V(T) = \{f(T) : f \in S(B(H))\}$

Proof.

If $0 \in V(T)$, then there exists an $f \in S(B(H))$ such that $f(T) = 0$ and $f(1) = 1 = \|f\|$. Now, $f(T) = 0$ implies that $\alpha + f(T) = \alpha$, or $f(\alpha I + T) = \alpha$. Taking the norm on both sides, then

$$|\alpha| = |f(\alpha I - T)| \leq \|f\| \|\alpha I - T\| = \|\alpha I - T\| .$$

Therefore $|\alpha| \leq \|\alpha I - T\|$ for all $\alpha \in \mathbf{C}$.

Conversely, assume that $|\alpha| \leq \|\alpha I - T\|$ for all $\alpha \in \mathbf{C}$. Define a positive linear functional f on the span of I and T by $f(\alpha T + \beta I) = \beta$. Then it can be shown that f is linear, positive, and that $f(1) = 1 = \|f\|$, $f(T) = 0$.

First, $f(T^*T) = \langle (T^*T)x, x \rangle = \langle Tx, Tx \rangle = \|Tx\|^2$. This shows that f is positive.

Also, $f(I) = \langle Ix, x \rangle = \langle x, x \rangle = \|x\|^2 = 1$.

Next, $1 = |f(I)| \leq \|f\| \|I\| = \|f\|$. That is $\|f\| \geq 1$.

Also, $\left|f(\alpha T+\beta I)\right| = \left|\langle(\alpha T+\beta I)x,x\rangle\right| \le \|(\alpha T+\beta I)x\|\|x\| \le \|\alpha T+\beta I\|$.

Therefore, $\left|f(\alpha T+\beta I)\right| \le \|\alpha T+\beta I\|$ and this shows that $\|f\| \le 1$. Therefore $\|f\| = 1$ and f is thus a state.

Finally

$$f(\alpha T+\beta I) = \langle(\alpha T+\beta I)x,x\rangle = \langle\alpha Tx,x\rangle + \langle\beta x,x\rangle = \alpha\langle Tx,x\rangle + \beta\langle x,x\rangle = \alpha f(T)+\beta.$$

This shows that $f(\alpha T+\beta I) = \alpha f(T)+\beta.$ But also, $f(\alpha T+\beta I) = \beta.$

Therefore $\alpha f(T)+\beta = \beta$ which shows that $f(T) = 0$, or $0 \in V(T)$.

The following results as found in [4] are immediate from the theorem 2.1.

Corollary 2.2.

$V(T)$ consists of those complex numbers λ for which $(T-\lambda) \perp I$.

Corollary 2.3.

$V(T)$ is equal to the convex hull of $\sigma(T)$ if and only if

$$\|T-\lambda\| = \left|\sigma(T-\lambda)\right|,$$

for all $\lambda \in \mathbf{C}$.

3. The Convexity of Algebraic Numerical Range

It will now be proved that the algebraic numerical range, $V(T)$, of $T \in B(H)$, is convex, and that it is contained in the closure of the ordinary numerical range $W(T)$.

Theorem 3.1.

The algebraic numerical range, $V(T)$, of $T \in B(H)$ is convex.

Proof.

It will be shown that if $\lambda_1, \lambda_2 \in V(T)$ and $\alpha \in (0,1)$, then

$$\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2 \in V(T).$$

Let $\lambda_1, \lambda_2 \in V(T)$. Then there exists states f_1, f_2 such that $\lambda_1 = f_1(T)$ and $\lambda_2 = f_2(T)$ where $T \in B(H)$ and $f_1(1) = 1 = \|f_1\|$, $f_2(1) = 1 = \|f_2\|$.

Define f on $B(H)$ by $f(T) = \alpha f_1(T) + (1-\alpha)f_2(T)$. It can be shown that f is a positive linear functional, and that $f(1) = 1 = \|f\|$.

First, it will be shown that f is a linear functional.

Let $\beta_1, \beta_2 \in \mathbf{C}$ and $T_1, T_2 \in B(H)$. Then

$$\begin{aligned}
& f(\beta_1T_1+\beta_2T_2) = \alpha f_1(\beta_1T_1+\beta_2T_2) + (1-\alpha)f_2(\beta_1T_1+\beta_2T_2) \\
& = \{\alpha f_1(\beta_1T_1) + (1-\alpha)f_2(\beta_1T_1)\} + \{\alpha f_1(\beta_2T_2) + (1-\alpha)f_2(\beta_2T_2)\} \\
& = \beta_1(\alpha f_1(T_1)) + \beta_1((1-\alpha)f_2(T_1)) + \beta_2(\alpha f_1(T_2)) + \beta_2((1-\alpha)f_2(T_2)) \\
& = \beta_1\{\alpha f_1(T_1) + (1-\alpha)f_2(T_1)\} + \beta_2\{\alpha f_1(T_2) + (1-\alpha)f_2(T_2)\} \\
& = \beta_1(f(T_1)) + \beta_2(f(T_2))
\end{aligned}$$

This shows that f is linear.

Next, it will be shown that f is positive.

Now, $f(T^*T) = \alpha f_1(T^*T) + (1-\alpha) f_2(T^*T) \geq 0$, since $f_1(T^*T) \geq 0$ and $f_2(T^*T) \geq 0$

Thus $f(T^*T) \geq 0$ for all $T \in B(H)$.

This shows that f is positive.

Next,

$f(I) = \alpha f_1(I) + (1-\alpha) f_2(I) = \alpha + (1-\alpha) = 1$.

Finally, it will be shown that $\|f\| = 1$.

Now,

$1 = |f(I)| \leq \|f\|\|I\| = \|f\|$. That is $\|f\| \geq 1$.

Also,

$$|f(I)| = |\alpha f_1(I) + (1-\alpha) f_2(I)| \leq |\alpha f_1(I)| + |(1-\alpha) f_2(I)|$$

$$\leq |\alpha||f_1(I)| + |1-\alpha||f_2(I)| = |\alpha| + |1-\alpha| = 1 = \|I\|.$$

That is, $\|f\| \leq 1$.

Therefore, $\|f\| = 1$.

Thus it is concluded that $\lambda \in V(T)$, hence $V(T)$ is convex.

Theorem 3.2.

$V(T) = \overline{W(T)}$

Proof.

Here, it will be shown that $V(T) \subseteq \overline{W(T)}$ and $V(T) \supseteq \overline{W(T)}$.

Agure [2], has proved that

(3.1) $$V(T) \subseteq \overline{W(T)}.$$

Thus it will only be proved that $V(T) \supseteq \overline{W(T)}$.

Let $\lambda \in \overline{W(T)}$. Then there is a sequence $\{x_n\}_{n\geq 1}$ of unit vectors in H such that

$$\lim_{n\to\infty} \langle Tx_n, x_n \rangle = \lambda \text{ and } \lim_{n\to\infty} \|Tx_n\| = \|T\|,$$

for all $T \in B(H)$.

Define a functional f on $B(H)$ by $f(T) = \lim_{n\to\infty} \langle Tx_n, x_n \rangle$, that is $f(T) = \lambda$.

It will be proved that f is a state on $B(H)$.

First, f is linear since if $S, T \in B(H)$ and $\alpha, \beta \in \mathbf{C}$ then,

$$f(\alpha S+\beta T)=\lim_{n\to\infty}\langle(\alpha S+\beta T)x_n,x_n\rangle=\lim_{n\to\infty}\langle\alpha Sx_n,x_n\rangle+\lim_{n\to\infty}\langle\beta Tx_n,x_n\rangle$$

$$=\alpha\lim_{n\to\infty}\langle Sx_n,x_n\rangle+\beta\lim_{n\to\infty}\langle Tx_n,x_n\rangle=\alpha f(S)+\beta f(T).$$

Also, f is positive since

$$f(T^*T)=\lim_{n\to\infty}\langle(T^*T)x_n,x_n\rangle=\lim_{n\to\infty}\langle Tx_n,Tx_n\rangle=\{\lim_{n\to\infty}\|Tx_n\|\}^2=\|T\|^2\geq 0.$$

Next, for $I\in B(H)$, $f(I)=\lim_{n\to\infty}\langle Ix_n,x_n\rangle=\lim_{n\to\infty}\langle x_n,x_n\rangle=\{\lim_{n\to\infty}\|x_n\|\}^2=1$

Finally, it will be shown that $\|f\|=1$.

Now, $|f(T)|=\left|\lim_{n\to\infty}\langle Tx_n,x_n\rangle\right|\leq\lim_{n\to\infty}\|Tx_n\|\lim_{n\to\infty}\|x_n\|=\|T\|$

This shows that $\|f\|\leq 1$.

Also, $1=\|f(I)\|\leq\|f\|\|I\|=\|f\|$. That is $\|f\|\geq 1$.

Therefore $\lambda\in V(T)$ and hence

(3.2) $$V(T)\supseteq\overline{W(T)}.$$

From equation 3.1 and equation 3.2, it is seen that $V(T)=\overline{W(T)}$.

4. FINITE OPERATORS

Williams has stated in theorem 4 in [4], that for an operator $T\in B(H)$, T is finite $\Leftrightarrow$, $\inf_S\|TS-ST-I\|=1$ for all $S\in B(H)\Leftrightarrow$ there exists a state f such that $f(TS)=f(ST)$ for all $S\in B(H)$.

These equivalent conditions will be proved here.

Theorem 4.1.

For an operator $T\in B(H)$, the following conditions are equivalent:

(i) $T\in B(H)$ is finite.

(ii) $\inf_S\|TS-ST-I\|=1$ for all $S\in B(H)$,

(iii) There exists a state f such that $f(TS)=f(ST)$ for all $S\in B(H)$.

Proof.

First it will be shown that $(i)\Rightarrow(ii)$.

From the definition of finite operators, if $T\in B(H)$ is finite, then $\|TS-ST-I\|\geq 1$ for all $S\in B(H)$. This is equivalent to $\inf_S\|TS-ST-I\|=1$.

Next, it will be shown that $(ii)\Rightarrow(iii)$.

If f is a state on $B(H)$, then, $|f(TS-ST-I)|\leq\|f\|\|TS-ST-I\|=\|TS-ST-I\|$ for all $S\in B(H)$. This shows that $\inf_S\|TS-ST-I\|=|f(TS-ST-I)|$.

Therefore $|f(TS-ST-I)|=1=\|I\|$. That is $f(TS-ST-I)=I$, or

$f(TS) - f(ST) - I = I$. This shows that $f(TS) = f(ST)$ for all $S \in B(H)$.

Finally, it will be shown $(iii) \Rightarrow (i)$.

Now, $f(TS) = f(ST)$ for all $S \in B(H)$ implies that $f(TS) - f(ST) = 0$ for all $S \in B(H)$ That is, $f(TS - ST) = 0$ for all $S \in B(H)$. This shows that $0 \in V(TS - ST)$ for all $S \in B(H)$, and hence $T \in B(H)$ is finite.

5. Conclusion

This paper has employed the inner products of vectors in a Hilbert space H as well as the properties of the states in $B(H)$ to show that zero is in the algebraic numerical range of an operator in $B(H)$ if and only if that operator is orthogonal to the identity operator. We have also shown that the algebraic numerical range of an operator in $B(H)$ is convex and is also equal to the closure of the spatial numerical range of that operator. One may also study more about the spectrum of operators in this C*-algebra.

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