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A Direct Approach to the Galois Procedure Concerning the Solvability of Polynomial Equations

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Abstract: *Many important developments, including valuable contributions in the domain of the Galois Theory have been accomplished the last decades. However, in several domains of mathematics important advances were obtained, and some of them could help bringing certain facilities also in this theory. Surely, certain fundamental conclusions of the Galois Theory can no longer be modified, but concerning its application in practice, several possibilities exist. For instance, the factorization of polynomials allows obtaining easily solutions for the equation group, roots included. This is one of the purposes of this work. At the same time, a certain quintic equation, apparently not solvable, but in fact having solutions with radicals, has been examined and we completed a case presented in an important conference further mentioned.*

Keywords: *group theory, factorization, solvability by radicals, Galois group, building up roots.*

1. Introduction

Many works have been devoted to the Galois Theory of equations, and many advances have been accomplished [1]-[12]. However, some mathematical procedures that previously were not too efficient, progressed much the last tenths years, but have not been used for improvements concerning the Galois Theory. It is just the subject of the present work.

We shall have in view two problems how to analyse the given equation, and how to proceed for constructing the Galois group of the given equation.

For analyzing a polynomial equation, several procedures have been used. Galois used the permutations completed with the root adjunction and group reduction, as recalled in [8]. We shall have in view the factorization of the given polynomial, extending to some extent the procedure of Maple 12. For the Galois group construction we shall use the procedure of Galois in the form given by Tignol [7, p. 238]. The solvability condition of Galois will be used as presented in [8], [9], the completion of Verriest included. We shall use the symbols and punctuation of the Maple software.

2. Factorization of the Polynomial Equation

In order to present the factorization, we shall refer first to the examples we used for other purposes in our preceding works. Let be the equation with rational coefficients of [3]:

$$f(x) = x^3 - 3x^2 + 1. \tag{1}$$

Unlike the usual cases, we shall factorize over the complex number field, otherwise the procedure we are using does not operate in many cases. The corresponding Maple operation will be used in the form:

$F := factor(f, complex);$

$$F := (x + 1.842085966) \cdot (x - 0.652736447) \cdot (x - 2.879385242); \tag{2}$$

Hence, we obtained almost immediately what with classical procedures requires many time.

However a certain inconveniency appears, we cannot precise the nature of the roots, except they are in this case all real. This circumstance is not grave because it cannot modify the automorphisms.

Now, we shall refer to the equation with rational coefficients of [8]:

$$f := x^3 - \frac{3}{2}x^2 + 1; \tag{3}$$

The corresponding Maple operation for factorization will be used in the same form as above:

$$F := factor(f, complex);$$

$$\begin{aligned} &F := (x + 0.676506988) \cdot (x - 1.088825349 + 0.5386519064I) \\ &\cdot (x - 1.088825349 - 0.5386519064I); \end{aligned} \tag{4}$$

Finally, we shall give an example presented by Xavier Caruso, on the occasion of the bicentenary of the birth of Galois, 24-28 October 2011, organizers being the Institut Henri Poincaré and the Société Mathématique de France, in Paris, for the symposium with the largest public, example designed just for this occasion. The example, an equation of fifth degree, is given below and was built up introducing the roots. We shall act conversely, for verifying the procedure we proposed, permitting to solve many equations of higher degree. The proposed equation is [12]:

$$g := x^5 - 10x^3 + 5x^2 + 10x + 1; \tag{5}$$

It is useful to note that equation was not directly proposed to solve, but has previously been constructed as we shall later explain. The corresponding Maple 12 operation for factorization will be:

$$G := factor(g, complex);$$

$$\begin{aligned} &G := (x + 3.259077382) \cdot (x + 0.7259775441) \cdot (x + 0.1069396117) \\ &\cdot (x - 1.562403693) \cdot (x - 2.52590845); \end{aligned} \tag{6}$$

3. Construction of the Galois Group

According to [1]-[3], we shall construct the Galois groups for the three equations above. With this procedure, the roots in form of complex conjugate, conjugate radical quantities or two real quantities with opposite signs may be interchanged. For the first equation we have:

Table 1. *Examples **a**, **b** and **c** of permutation group of a polynomial (equation, function).*

Function	$f := x^3 - \frac{3}{2}x^2 + 1$ **(a)**	$f := x^3 - 3x^2 + 1$ **(b)**
Group after Maple 12	"3T2", {"S(3)"}, "-", 6, {"(1 3)", "(2 3)"}	"3T1", {"A(3)"}, "+", 3, {"(1 2 3)"}
Group according to Galois	$\sigma_{id} = \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{pmatrix}$	$\sigma_{id} = \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{pmatrix}$
Roots	$1.08882534 + 0.5386519I$ $1.08882534 - 0.5386519I$ -0.6776506	$0.6527036 \quad 2.8793852 \quad -0.5320888$
Function	$g := x^5 - 10x^3 + 5x^2 + 10x + 1.$ **(c)**	
Group after Maple 12	"ST1",{"C(5)"}, "+",5, {"(1 2 3 4 5)"}	
Group according to Galois	$\sigma_{id} = \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \\ 4 \to 4 \\ 5 \to 5 \end{pmatrix}$ $\sigma_1 = \begin{pmatrix} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \\ 4 \to 3 \\ 5 \to 5 \end{pmatrix}$	
Roots	$3.59077382 \quad 0.72597754 \quad 0.1069396117 \quad -1.562403693 \quad -2.52590845$	

The sign of the equation group is depending on the type of transpositions. In the cases above, where no transposition occurs, the sign will be plus, while in the other, it will be minus. A simplification in table was made relatively to [9].

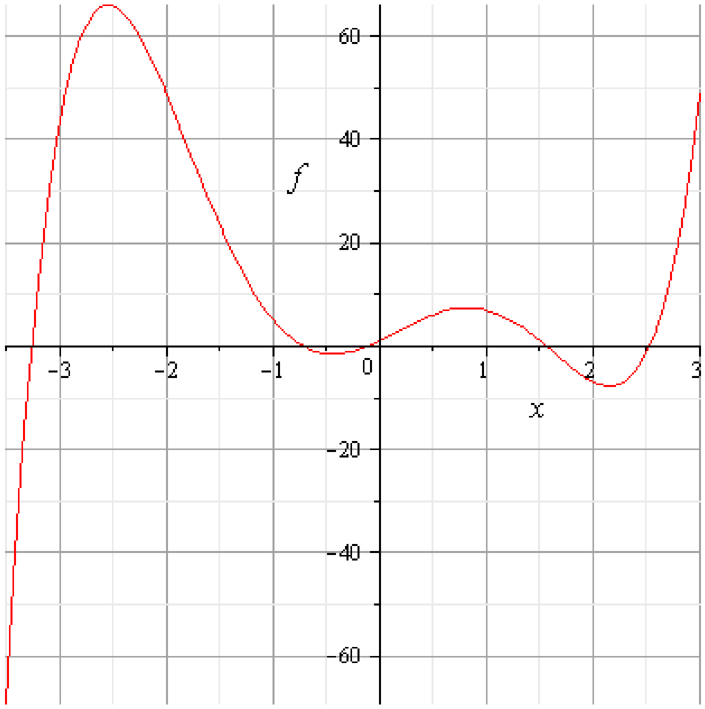

Fig. 1. The curve of a quintic polynomial built up by Xavier Caruso.

4. Establishing the Roots

4.1. The starting values

The curve of Fig. 1 was constructed starting from certain expressions of the roots. Now, we shall also construct the concerned polynomial, but in a certain different manner, for describing carefully all steps and making evident several possibilities not given in the original presentation.
We shall start choosing the determination of an angle, say, as in original form:

$$\mathrm{arc} := 2\cdot\frac{\mathrm{Pi}}{25} \tag{7}$$

what means the 25-th part of a circle or the fifth part of the angle of a pentagon. It is worth noting that the trigonometric functions of this angle can be expressed by radical expressions containing $\sqrt{5}$.
Having in view to establish a quintic polynomial equation, we shall search for the component quantities of the roots:

$$\zeta := \exp\left(\frac{2\cdot\mathrm{Pi}}{25}\cdot I+\frac{2\cdot\mathrm{Pi}}{5}\cdot I\cdot k\right),\quad k\in[1,\quad 5]; \tag{7 a}$$

$$\zeta_k := \exp\left(\frac{2\cdot\mathrm{Pi}+10\cdot\mathrm{Pi}\cdot k}{25}\cdot I\right),\quad k\in[1,\quad 5]; \tag{7 b}$$

where the last term of the round parenthesis represents the five quantities resulted as the roots of the unity.

4.2. The four components of each of the five roots

The five values may be written as follows:

$$\zeta_0 := \exp\left(\frac{2\cdot\mathrm{Pi}}{25}\cdot I\right);\qquad \zeta_1 := \exp\left(\frac{12\cdot\mathrm{Pi}}{25}\cdot I\right);\qquad \zeta_2 := \exp\left(\frac{22\cdot\mathrm{Pi}}{25}\cdot I\right);$$
$$\zeta_3 := \exp\left(\frac{32\cdot\mathrm{Pi}}{25}\cdot I\right);\qquad \zeta_4 := \exp\left(\frac{42\cdot\mathrm{Pi}}{25}\cdot I\right); \tag{8 a-e}$$

For establishing an equation of fifth degree, the first step can be to search for the roots of the equation be real ones. We can write

$$x_0 := \zeta_0 + (\zeta_0)^p + (\zeta_0)^q + (\zeta_0)^r; \tag{9}$$

In the original paper, there is not justified why have been chosen just four terms. We can argue that a simple manner for the imaginary part vanishes, is by associating the first and last term and the other two, separately.

In the same presentation, its author proposed the exponents: $p = 1, q = 7, q = 18, r = 24$, but did not give other reason. We examined the case, and concluded that there are several solutions, at any rate the smaller the exponents, the better will be, to some extent, the results, the number of computations being smaller. We chose the exponents: $p = 1, q = 7, q = -7, r = -1$.

5. The Five Roots Established for the Equation

Finally, we obtained the expressions:

$$\begin{aligned}
x_0 &:= \zeta_0 + (\zeta_0)^7 + (\zeta_0)^{-7} + (\zeta_0)^{-1};\\
x_1 &:= \zeta_1 + (\zeta_1)^7 + (\zeta_1)^{-7} + (\zeta_1)^{-1};\\
x_2 &:= \zeta_2 + (\zeta_2)^7 + (\zeta_2)^{-7} + (\zeta_2)^{-1};\\
x_3 &:= \zeta_3 + (\zeta_3)^7 + (\zeta_3)^{-7} + (\zeta_3)^{-1};\\
x_4 &:= \zeta_4 + (\zeta_4)^7 + (\zeta_4)^{-7} + (\zeta_4)^{-1};
\end{aligned} \tag{10 a-e}$$

For constructing the polynomial equation of fifth degree, it is necessary to compute the coefficients and should obtain rational or whole coefficients.

The numerical results yield:

$$\begin{aligned}
x_0 &= 1.562403692 - 1.10^{-10}\, I,\\
x_1 &= -0.7259775411 + 5.10^{-10}\, I,\\
x_2 &= -0.1069396180 + 0.\, I,\\
x_3 &= -0.3259077382 + 0.\, I,\\
x_4 &= -0.2529590846 - 6.10^{-10}\, I,
\end{aligned} \tag{11 a-e}$$

and we can see that the imaginary parts vanish.

6. The Coefficients of the Equation Obtained from the Roots

We shall calculate the coefficients of the quintic polynomial having the leading factor equal to unity:

$$Y := x^5 + a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e; \tag{12}$$

A simple calculation yields:

$$\begin{aligned}
Y &:= \exp\text{and}((x + 3.259077382)\cdot(x + 0.7259775441)\cdot(x + 0.1069396117)\\
&\cdot(x - 1.562403693)\cdot(x - 2.52590845));
\end{aligned} \tag{13}$$

and

$$Y = x^5 - 10.000000\,x^3 + 5.000000\,x^2 + 10.000000\,x + 1.000000. \tag{14}$$

For a verification of every coefficient, we have:

$$a := -\sum_{i=0}^{n-1} x_i; \tag{15}$$

$$b := \sum_{i=0}^{n-2} x_i \cdot \left(\sum_{j=i+1}^{n-1} x_j\right); \tag{16}$$

$$c := \sum_{i=0}^{n-3} x_i \cdot \left(\sum_{j=i+1}^{n-2} x_j \cdot \left(\sum_{k=j+1}^{n-1} x_k \right) \right); \tag{17}$$

$$d := \sum_{i=0}^{n-4} x_i \cdot \left(\sum_{j=i+1}^{n-3} x_j \cdot \left(\sum_{k=j+1}^{n-2} x_k \cdot \left(\sum_{l=k+1}^{n-1} x_k \right) \right) \right); \tag{18}$$

$$e := x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4 ; \tag{19}$$

There follows

$$a = 3.10^{-9}, \quad b = -10.000000, \quad c = 4.9999999, \quad d = 10.0000000, \quad e = 1.000000 . \tag{20}$$

Therefore, the whole coefficient numbers are:

$$a = 0, \quad b = -10, \quad c = 5, \quad d = 10, \quad e = 1, \tag{21}$$

and we found the proposed equation.

Remark. If for the polynomials of Table 1, we used for factorization the Maple command $factor(f, radical)$, having in view that for the case **a** or **b** a radical formula exists, the answer would be *error, in factor, 2nd argument, radical, is not a valid algebraic extension*. Another interesting circumstance is that in the result of the mentioned formula, in these cases, the real and imaginary parts cannot be directly separated, but certain previous transformations and the application of de Moivre formula are required. Except some small differences, the situation is alike for case **c**. A mention in literature was not found.

We may add that if the obtained roots, denoted as *results* belong to a solvable equation, their expressions in radical form, can be obtained by a MAPLE 12 command as: $convert(result, radical)$. The only difficulty we encountered was the relatively large length of expressions due to the number of the occurring radical terms.

7. The Solution with Radicals of an Equation of Fifth Degree

For this problem, we shall consider the quintic polynomial equation of above. For this purpose, we shall first look for the solution of the equation, using the Maple 12 formula:

$$Y := x^5 - 10 \cdot x^3 + 5 \cdot x^2 + 10 \cdot x + 1; \tag{22}$$

$$roots = solve(Y); \tag{23}$$

which yields:

$$\begin{aligned}
&RootOf(_z^5 - 10_z^3 + 5_z^2 + 10_z + 1, index = 1),\\
&RootOf(_z^5 - 10_z^3 + 5_z^2 + 10_z + 1, index = 2),\\
&RootOf(_z^5 - 10_z^3 + 5_z^2 + 10_z + 1, index = 3),\\
&RootOf(_z^5 - 10_z^3 + 5_z^2 + 10_z + 1, index = 4),\\
&RootOf(_z^5 - 10_z^3 + 5_z^2 + 10_z + 1, index = 5) .
\end{aligned} \tag{24 a-e}$$

For obtaining the roots in radical form, we calculate as follows:

$$R_1 := RootOf(\_z^5 - 10\_z^3 + 5\_z^2 + 10\_z + 1, index = 1); \tag{25 a}$$

$$x_1 = convert(R_1, radical); \tag{25 b}$$

and there follows

$$
\begin{aligned}
x_1 &= \frac{1}{5}\left(-\frac{3125}{4}+\frac{3125}{4}\sqrt{5}-\frac{125}{4}\sqrt{-125+10\sqrt{5}}-\frac{\frac{48125}{4}I\sqrt{5}}{\sqrt{125-10\sqrt{5}}}\right)^{1/5} \\
&+\frac{1}{5}\cdot\frac{-\frac{625}{4}-\frac{625}{4}\sqrt{5}-\frac{175}{4}\sqrt{-125+10\sqrt{5}}+\frac{1375}{4}I\sqrt{5}}{\left(-\frac{3125}{4}+\frac{3125}{4}\sqrt{5}-\frac{125}{4}\sqrt{-125+10\sqrt{5}}-\frac{\frac{48125}{4}I\sqrt{5}}{\sqrt{125-10\sqrt{5}}}\right)^{3/5}} \\
&+\frac{1}{5}\cdot\frac{-\frac{125}{4}+\frac{125}{4}\sqrt{5}-\frac{5}{4}\sqrt{-125+10\sqrt{5}}+\frac{1925}{\sqrt{125-10\sqrt{5}}}I\sqrt{5}}{\left(-\frac{3125}{4}+\frac{3125}{4}\sqrt{5}-\frac{125}{4}\sqrt{-125+10\sqrt{5}}-\frac{\frac{48125}{4}I\sqrt{5}}{\sqrt{125-10\sqrt{5}}}\right)^{2/5}} \qquad (26) \\
&+\frac{5}{\left(-\frac{3125}{4}+\frac{3125}{4}\sqrt{5}-\frac{125}{4}\sqrt{-125+10\sqrt{5}}-\frac{\frac{48125}{4}I\sqrt{5}}{\sqrt{125-10\sqrt{5}}}\right)^{1/5}}.
\end{aligned}
$$

Similarly, the other four roots can be calculated. The roots being known, the equation group, according to Galois, can be built up as previously shown. If the equation was not solvable, the command (25 b) could not work, but it would repeat the last command. Such as examples could be the unsolvable equations x^5-10x^3+1 and x^5-3x^3+1. However, even in this case, the Maple 12 command *factor* works.

8. The Solvability of Polynomial Equations

From the form of the roots, it results that the last considered equation is solvable by radicals as expected, although the equations of a degree greater than 4, do not satisfy the general condition for solvability. We have examined especially the finite simple groups. These groups may be classified completely into several classes as detailed in [8], [13]. We shall confine ourselves only to: 1. Symmetric groups S_n and 2. Alternating groups A_n.

The symmetric group of permutations leaves unchanged any relation among the roots, in the number field, for both odd and even permutations. The latter ones have the same effect for the discriminant, being similar to the equation group. We recall that the set of even permutations of a symmetric group of any degree n represents a subgroup, because the product of two such permutations yields also an even permutation. As already mentioned, the odd permutations do not satisfy this condition, being called a complex adjoined. The number of permutations of the mentioned subgroup should be $\frac{n!}{2}$.

Therefore, according to Galois Theory, the sequence of composition factors includes, at its end, the ratio of A_n and E, except the case of $n=4$. It may be mentioned that in the case of a quartic, $n=4$, the permutation order (number of permutations) namely 12 occurs as the second permutation, and no reason to move it at the end. In the general case, we can see that for $n>4$, this ratio cannot be a prime number. It follows that in the general case, any group of degree $n>4$ is not metacyclic. However, as in the last case examined in this work, this conclusion is not valid in every case, the form of the roots assuring a solvable form of the polynomial.

9. Conclusion

A direct approach for the analysis of a Galois group of a polynomial equation has been developed which may permit an efficient procedure for many practical cases. This procedure is based on the factorization, under certain rules, of the given polynomials (equations). Among the given cases, an equation of fifth degree, presented to the bicentenary of Évariste Galois, by X. Caruso, has been analysed and additional results have been carried out.

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Andrei Costin Nicolaide was born on the 1st of September 1933 in Bucharest. He received the degree of Electrical Engineer with honours, from Technical Institute of Craiova, Faculty of Electrotechnics (1956); Doctor of Engineering and Doctor of Sciences (Polytechnic Institute of Bucharest, in 1962 and 1974, respectively). Full professor at the Transilvania University of Brasov (1969-2003), consulting professor since 2004. His scientific activity includes field computation by conformal transformation and numerical methods, and Special and General Theoryof Relativity. He is a regular member of the Academy of Technical Sciences of Romania.