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Some constructions of A-Optimum, Variance and Efficiency Balance Weighing Designs

Rashmi Awad
School of Statistics,
Devi Ahilya University,
Indore, India.
awad.rashmi@gmail.com

Dr. Shakti Banerjee
Professor and Head, School of Statistics,
Devi Ahilya University,
Indore, India.
shaktibn@yahoo.com

Abstract: *Some new construction methods of the optimum chemical balance weighing designs are proposed, which are based on the incidence matrices of the known balanced incomplete block designs, balanced bipartite block designs and ternary balanced block designs. Also the conditions under which the constructed chemical balance weighing designs become A-optimal are also been given.*

Keywords: *Balance incomplete block design, balanced bipartite block design, ternary balanced block design, variance balanced design, efficiency balanced design, weighing design, chemical balance weighing design, optimum chemical balance weighing design, A-optimal chemical balance weighing design.*

1. INTRODUCTION

In recent years, the new methods of constructing the optimum chemical balance weighing designs and a lower bound for the variance of each of the estimated weights from this chemical balance weighing design were obtained and a necessary and sufficient condition for this lower bound to be attained was proposed in the literature; see [1]-[8]. These constructions were based on the incidence matrices of balanced incomplete block designs, balanced bipartite block designs, ternary balanced block designs and group divisible designs.

Awad et al. [9]-[12] gave the construction methods of obtaining optimum chemical balance weighing designs using the incidence matrices of symmetric balanced incomplete block designs and some pairwise balanced designs were also been obtained which were efficiency as well as variance balanced. In this paper we now propose new construction methods of obtaining optimum chemical balance weighing designs using the incidence matrices of balanced incomplete block design, balanced bipartite block design, ternary balanced block design and some more pairwise efficiency as well as variance balanced designs are proposed. Also we present the conditions under which the chemical balance weighing designs constructed by new construction methods leading to the A-optimal designs.

Let us consider ν treatments arranged in b blocks, such that the j^{th} block contains k_j experimental units and the i^{th} treatment appears r_i times in the entire design, i = 1,2,…….,ν; j = 1,2,……,b. For any block design there exist a incidence matrix N = $[n_{ij}]$ of order ν x b, where n_{ij} denotes the number of experiment units in the j^{th} block getting the i^{th} treatment. When n_{ij} = 1 or 0 ∀ i and j, the design is said to be binary. Otherwise it is said to be nonbinary. In this paper we consider binary block designs only. The following additional notations are used $\underline{k}$ = $[k_1\ k_2\ldots..k_b]$' is the column vector of block sizes, $\underline{r}$ = $[r_1\ r_2\ldots..r_v]$' is the column vector of treatment replication, K_{bxb} = diag $[k_1\ k_2\ldots..k_b]$, R_{vxv} = diag $[r_1\ r_2\ldots..r_v]$, $\sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j = n$ is the total number of experimental units, with this $N1_b = \underline{r}$ and $N'1_v = \underline{k}$, Where 1_a is the a x 1 vector of ones.

An equi-replicate, equi-block sized, incomplete design, which is also balanced in the sense given above is called balanced incomplete block design, which is an arrangement of ν symbols (treatments) into b sets (blocks) each containing *k* (< ν) distinct symbols, such that any pair of distinct symbols occurs in exactly λ sets. Then it is easy to see that each treatment occurs in *r* (>λ) sets. ν, b, r, k, λ are

called parameters of the BIBD and the parameters satisfies the relations $\nu r = bk$, $r(k-1) = \lambda(\nu - 1)$ and $b \geq \nu$ (Fisher's Inequality).

Though there have been balanced designs in various senses (see [13] [14]). We will consider a balanced design of the following type.

A block design is called variance balanced if and only if

1. it permits the estimation of all normalized treatment contrasts with the same variance (see[15])

2. if the information matrix for treatment effects $C = R - NK^{-1}N'$ satisfies $C = \mu [I_v - (1/\nu) 1_v 1_v']$

where μ is the unique nonzero eigen value of the matrix C with the multiplicity $(v - 1)$, I_v is the $v \times v$ identity matrix.

A block design is called efficiency balanced if

1. every contrast of treatment effects is estimated through the design with the same efficiency factor.

2. $M_o = R^{-1}NK^{-1}N' - (1/n) 1_v r' = \psi (I_v - (1/n) 1_v r')$; See [16]

and since $M_o S = \psi S$, where ψ is the unique non zero eigen value of M_o with multiplicity $(\nu-1)$. For the EB block design N, the information matrix C is given as $C = (1- \psi)(R - (1/n) r r')$; see [17].

A block design is said to be pairwise balanced if $\sum_{j=1}^{b} n_{ij} n_{i'j} = \Lambda$ (a constant) for all i, i', $i \neq i'$ and a pairwise balanced block design is said to be binary if $n_{ij} = 0$ or 1 only, for all i, j and it has parameters ν, b, r, k, Λ (= λ, say) [in this case, when $\underline{r} = r1_v$ and $\underline{k} = k1_b$, it is a BIB design with parameters ν, b, r, k, λ].

A balanced bipartite block design is an arrangement of v treatments into b blocks such that each block containing k distinct treatments is divided into two subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks. Every pair of treatments with different subblocks appears together in λ_1 blocks and every pair of treatments with the same subblock appears together in λ_2 blocks. The integers v, b, r, k_1, k_2, λ_1, λ_2 are called the parameters of the balanced bipartite block design. Let N^* be the incidence matrix of this design. The parameters are not all independent and they are related by the following identities

$$\nu r = b k$$

$$b = \lambda_1 \nu (\nu-1) / (2k_1 k_2)$$

$$\lambda_2 = \lambda_1 [k_1 (k_1-1) + k_2 (k_2-1)] / (2k_1 k_2) \qquad (1)$$

$$r = \lambda_1 k (\nu-1) / (2k_1 k_2)$$

$$N^*N^{*\prime} = (r - \lambda_1 - \lambda_2) I_v + (\lambda_1 + \lambda_2) 1_v 1_v'$$

where 1_v is v x 1 vector with elements equal to 1 everywhere.

If in the balanced bipartite block design the number of objects in the first sub block is not equal to the number of objects in the second sub block ($k_1 \neq k_2$) then each object exists in r_1 blocks in the first sub block and in r_2 blocks in the second sub block, $r_1 + r_2 = r$.

Then

$$r_1 = \frac{\lambda_1 (\nu - 1)}{2 k_2} \quad \text{and} \quad r_2 = \frac{\lambda_1 (\nu - 1)}{2 k_1}$$

A ternary balanced block design is a design consisting of b blocks, each of size k, chosen from a set of size v in such a way that each of the v elements occurs r times all together and 0,1 or 2 times in each block and each of the $\binom{\nu}{2}$ distinct pairs of elements occurs λ times. Any ternary balanced block design is regular, that is, each element occurs singly in ρ_1 blocks and is repeated in ρ_2 blocks, where ρ_1 and ρ_2 are constant for the design. Accordingly we write the parameters of ternary balanced block design in the form $v, b, r, k, \lambda, \rho_1, \rho_2$. Let N be the incidence matrix of ternary balanced block design. Here the parameters are related by the following identities

$vr = bk$,

$r = \rho_1 + 2\rho_2$

$\lambda (v-1) = \rho_1 (k-1) + 2\rho_2 (k-2) = r (k-1) - 2\rho_2$

$$N N' = (\rho_1 + 4\rho_2 - \lambda) I_v + \lambda 1_v 1_v' = (r + 2\rho_2 - \lambda) I_v + \lambda 1_v 1_v' \quad (2)$$

Weighing designs consists of n groupings of the p objects and suppose we want to determine the individual weights of p objects. We can fit the results into the general linear model

$$\underline{Y} = X \underline{w} + \underline{e} \quad (3)$$

where $\underline{Y}$ is an n×1 random column vector of the observed weights, $\underline{w}$ is the p×1 column vector representing the unknown weights of objects and $\underline{e}$ is an n×1 random column vector of errors such that $E(\underline{e}) = 0_n$ and $E(\underline{e}\,\underline{e}') = \sigma^2 I_n$. $X = (x_{ij})$, (i=1,2,. . . ,n; j=1,2,. . . ,p) is a n×p matrix of known quantities. The elements of matrix X takes the values as

$$x_{ij} = \begin{cases} +1 & \text{if the } j^{th} \text{ object is placed in the left pan in the } i^{th} \text{ weighing,} \\ -1 & \text{if the } j^{th} \text{ object is placed in the right pan in the } i^{th} \text{ weighing} \\ 0 & \text{if the } j^{th} \text{ object is not weighted in the } i^{th} \text{ weighing} \end{cases}$$

The normal equations estimating w are of the form

$$X'X \underline{\hat{w}} = X'\underline{Y} \quad (4)$$

where $\underline{\hat{w}}$ is the vector of the weights estimated by the least squares method.

The matrix X is called the design matrix. A weighing design is said to be singular or nonsingular, depending on whether the matrix $X'X$ is singular or nonsingular, respectively. It is obvious that the matrix $X'X$ is nonsingular if and only if the matrix X is of full column rank (=p). Now, if X is of full rank, that is, when $X'X$ is nonsingular, the least squares estimate of $\underline{w}$ is given by

$$\underline{\hat{w}} = (X'X)^{-1} X'\underline{Y} \quad (5)$$

and the variance-covariance matrix of $\underline{\hat{w}}$ is

$$Var(\underline{\hat{w}}) = \sigma^2 (X'X)^{-1} \quad (6)$$

When the objects are placed on two pans in a chemical balance, we shall call the weighings two pan weighing and the design is known as two pan design or chemical balance weighing design. In chemical balance weighing design, the elements of design matrix $X = \{x_{ij}\}$ takes the values as +1 if the j^{th} object is placed in the left pan in the i^{th} weighing, -1 if the j^{th} object is placed in the right pan in the i^{th} weighing and 0 if the j^{th} object is not weighted in the i^{th} weighing.

Hotelling has shown that if n weighing operations are to determine the weights of p = n objects, the minimum attainable variance for each of the estimated weights in this case is σ^2/n and proved the theorem that each of the variance of the estimated weights attains the minimum if and only if $X'X = nI_p$ (See[18]).

2. VARIANCE LIMIT OF ESTIMATED WEIGHTS

Let X be an $n \times p$ matrix of rank p of a chemical balance weighing design and let m_j be the number of times in which j^{th} object is weighed, j=1,2,…., p (i.e. the m_j be the number of elements equal to -1 and 1 in j^{th} column of matrix X). Then Ceranka et al. (see [1]) proved that for any n x p matrix X, of a nonsingular chemical balance weighing design, in which maximum number of elements equal to -1 and 1 in columns is equal to m, where $m = \max\{m_1, m_2, \ldots\ldots, m_p\}$. Then each of the variances of the estimated weights attains the minimum if and only if

$$X'X = m I_p \quad (7)$$

Also a nonsingular chemical balance weighing design is said to be optimal for the estimating individual weights of objects if the variances of their estimators attain the lower bound given by,

$$Var(\hat{w}) = \frac{\sigma^2}{m}, j = 1,2,.....p \tag{8}$$

3. CONSTRUCTION OF DESIGN MATRIX WEIGHTS

Let N_1 be the incidence matrix of the balanced incomplete block design with the parameters v, b_1, r_1, k_1, λ_1 and in $(2N_1' - J_{b_1 \times v})$ matrix r_1 elements equal to +1, (v - r_1) elements equal to -1. N_2^* be the incidence matrix of balanced bipartite block design with the parameters v, b_2, r_2, k_{12}, k_{22}, λ_{12}, λ_{22} . Now, the N_2 can be obtained from N_2^* by replacing the k_{12} elements equal to +1 of each column which correspond to the elements belonging to the first sub block by -1. Thus each column of N_2 will contain k_{12} elements equal to -1, k_{22} elements equal to 1 and v - k_{12}- k_{22} elements equal to 0. Also N_3 denotes the incidence matrix of ternary balanced block design with the parameters v, b_3, r_3, k_3, λ_3, ρ_{13}, ρ_{23} and matrix ($N_3' - J_{b_3 \times v}$) contains ρ_{13} elements equal to -1, ρ_{23} elements equal to +1 and v-ρ_{13}-ρ_{23} elements equal to zero.

Now, we define the matrix X of chemical balance weighing design as

$$X = \begin{bmatrix} 2N_1' - J_{b_1 \times v} \\ N_2' \\ N_2' \\ N_3' - J_{b_3 \times v} \end{bmatrix} or \begin{bmatrix} 2N_1' - J_{b_1 \times v} \\ N_2' \\ -N_2' \\ N_3' - J_{b_3 \times v} \end{bmatrix} \tag{9}$$

Under the present construction scheme, we have $n=b_1+2\ b_2+\ b_3$ and $p = v$. Clearly such a design implies that each object is weighted $m = b_1+ 2\ r_2 + (\rho_{13} + \rho_{23})$ times in $n=b_1+2\ b_2+\ b_3$ weighing operations.

Lemma 3.1. A design given by X of the form (9) is non singular if and only if $v \neq 2k_1$, $k_{12} \neq k_{22}$ and $v \neq k_3$.

Proof. For the design matrix X given by (9), we have

$$X'X = \left[4(r_1 - \lambda_1) + 2(r_2 - \lambda_{22} + \lambda_{12}) + (r_3 + 2\rho_{23} - \lambda_3)\right] I_v + \left\{4\lambda_1 - 4r_1 + b_1 + 2(\lambda_{22} - \lambda_{12}) + \lambda_3 - 2r_3 + b_3\right\} J_{vv} \tag{10}$$

and $$\left|X'X\right| = \left[4(r_1 - \lambda_1) + 2(r_2 - \lambda_{22} + \lambda_{12}) + (r_3 + 2\rho_{23} - \lambda_3)\right]^{v-1}$$

$$\times \left[\frac{r_1}{k_1}(v - 2k_1)^2 + 2\left\{\frac{(v-1)\lambda_{12}}{2k_{12}k_{22}}(k_{12} - k_{22})^2\right\} + \frac{r_3(v - k_3)^2}{k_3}\right] \tag{11}$$

Since $4(r_1 - \lambda_1) + 2(r_2 - \lambda_{22} + \lambda_{12}) + (r_3 + 2\rho_{23} - \lambda_3)$ is positive and thus the determinant (11) is equal to zero if and only if

$v = 2k_1$,

$k_{12} = k_{22}$

and $v = k_3$

So the lemma is proved.

Theorem 3.2. The non-singular chemical balance weighing design with matrix X given by (9) is optimal if and only if $4(\lambda_1 - r_1) + b_1 + 2(\lambda_{22} - \lambda_{12}) + \lambda_3 - 2r_3 + b_3 = 0$. (12)

Proof. From the conditions (7) and (10) it follows that a chemical balance weighing design is optimal if and only if the condition (12) holds. Hence the theorem.

If the chemical balance weighing design given by matrix X of the form (9) is optimal then

$$Var\left(\hat{w}_j\right) = \frac{\sigma^2}{b_1 + 2r_2 + (\rho_{13} + \rho_{23})} \quad ; \quad j = 1,2,....., p$$

Example 3.3. Consider a balanced incomplete block design with parameters $v = 5$, $b_1 = 5$, $r_1 = 4$, $k_1 = 4$, $\lambda_1 = 3$; balanced bipartite design with parameters BIB design with parameters $v = 5$, $b_2 = 5$, $r_2 = 4$, $k_{12} = 2$, $k_{22} = 2$, $\lambda_{12}=2$, $\lambda_{22} = 1$ and balanced ternary design with parameters $v = 5$, $b_3 = 15$, $r_3 = 9$, $k_3 = 3$, $\lambda_3 = 1$, $\rho_{13} = 7$, $\rho_{23}=1$.

Theorem 3.2 yields a design matrix X of optimum chemical balance weighing design as

$$X = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ - & - & - & - & - \\ -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ - & - & - & - & - \\ -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ - & - & - & - & - \\ 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 \\ -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ - & - & - & - & - \\ -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ - & - & - & - & - \\ 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 1 \\ - & - & - & - & - \\ 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 \\ -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Clearly such a design implies that each object is weighted m = 21 times in n = 30 weighing operations and $Var(\hat{w}_j) = \sigma^2/21$ for each j = 1,2, ... ,5.

4. A-OPTIMALITY OF CHEMICAL BALANCE WEIGHING DESIGN

Some problems related to the optimality of chemical balance weighing designs were considered in the literature; see [19], [20], [21]. Wong and Masaro [22], [23] gave the lower bound for tr $[(X'X)^{-1}]$ and some construction methods of the A-optimal chemical balance weighing designs.

Let X be an n × p design matrix of a chemical balance weighing design. Then the following results from Ceranka et al. [24] give the lower bound for tr $[(X'X)^{-1}]$.

Theorem 4.1. For any nonsingular chemical balance weighing design with the design matrix $X = (x_{ij})$ we have

$$\text{tr}\,[(X'X)^{-1}] \geq \frac{p^2}{q.n} \tag{13}$$

where $q = \max(q_1, q_2, ..., q_n)$, $q_i = \sum_{j=1}^{p} x_{ij}^2$, i = 1, 2, .., n.

The case when q = p; we get the inequality given in Wong and Masaro [22].

Definition 4.2. Any nonsingular chemical balance weighing design with the design matrix $X = (x_{ij})$ is said to be A-optimal if

$$tr[(X'X)^{-1}] = \frac{p^2}{q.n}. \tag{14}$$

Theorem 4.3. Any nonsingular chemical balance weighing design with the design matrix $X = (x_{ij})$ is A-optimal if and only if

$$X'X = \frac{q.n}{p} I_p. \tag{15}$$

5. CHECKING THE A-OPTIMALITY

For the construction Method of chemical balance weighing design; the Lemma 3.1 proven above gave the necessary condition for the design matrix X of the form (9) to be non-singular.

Theorem 5.1. The non-singular chemical balance weighing design with matrix X given by (9) is A-optimal if and only if

$$4\lambda_1 - 4r_1 + b_1 + 2(\lambda_{22} - \lambda_{12}) + \lambda_3 - 2r_3 + b_3 = 0 \tag{16}$$

and

$$\left[4(r_1 - \lambda_1) + 2(r_2 - \lambda_{22} + \lambda_{12}) + (b_3 - \rho_{13})\right] = \frac{q[b_1 + 2b_2 + b_3]}{v} \tag{17}$$

Proof. For the design matrix X given in (9) we have

$$X'X = \left[4(r_1 - \lambda_1) + 2(r_2 - \lambda_{22} + \lambda_{12}) + (r_3 + 2\rho_{23} - \lambda_3)\right] I_v + \left\{4\lambda_1 - 4r_1 + b_1 + 2(\lambda_{22} - \lambda_{12}) + \lambda_3 - 2r_3 + b_3\right\} J_{vv}$$

and $$X'X = \frac{q[b_1 + 2b_2 + b_3]}{v} I_v$$

Comparing these two equalities we get

$$4\lambda_1 - 4r_1 + b_1 + 2(\lambda_{22} - \lambda_{12}) + \lambda_3 - 2r_3 + b_3 = 0$$

and

$$\left[4(r_1 - \lambda_1) + 2(r_2 - \lambda_{22} + \lambda_{12}) + (r_3 + 2\rho_{23} - \lambda_3)\right] = \frac{q[b_1 + 2b_2 + b_3]}{v}$$

$$\Rightarrow \quad \left[4(r_1 - \lambda_1) + 2(r_2 - \lambda_{22} + \lambda_{12}) + (b_3 - \rho_{13})\right] = \frac{q[b_1 + 2b_2 + b_3]}{v}$$

If (16) is satisfied then we get the condition (17) from the last equation. Hence the theorem.

Note: There are only some designs with certain parameters which satisfy A-optimality condition given in (17); (see [24]).

6. CONSTRUCTION OF PAIRWISE DESIGN

If N_1 be the incidence matrix of the balanced incomplete block design with the parameters $v, b_1, r_1, k_1, \lambda_1$. N_2^* be the incidence matrix of balanced bipartite block design with the parameters $v, b_2, r_2, k_{12}, k_{22}, \lambda_{12}, \lambda_{22}$ and N_3 denotes the incidence matrix of ternary balanced block design with the parameters $v, b_3, r_3, k_3, \lambda_3, \rho_{13}, \rho_{23}$. Then the structure

$$N^* = \begin{bmatrix} N_1 & \vdots & N_2^* & \vdots & N_2^* & \vdots & N_3 \end{bmatrix} \tag{18}$$

form a pairwise VB and EB design.

Theorem 6.1: The structure defined in equation (18); implies the existence of a pairwise VB and EB design. *D** with parameters

$v^* = v,\ b^* = b_1 + 2\,b_2 + b_3,\ r^* = r_1 + 2\,r_2 + r_3,\ k^* = \begin{bmatrix} k_1 1'_{b_1} & k_2 1'_{b2} & k_2 1'_{b2} & k_3 1'_{b_3} \end{bmatrix},\ \lambda^* = \lambda_1 + 2(\lambda_{12} + \lambda_{22}) + \lambda_3,$

$$\mu^* = v\left(\frac{\lambda_1}{k_1} + \frac{2\lambda_2}{k_2} + \frac{\lambda_3}{k_3}\right) \quad \text{and} \quad \psi^* = 1 - \frac{v}{(r_1 + 2r_2 + r_3)}\left(\frac{\lambda_1}{k_1} + \frac{2\lambda_2}{k_2} + \frac{\lambda_3}{k_3}\right).$$

Proof : In BIB design D_1 with the parameters (v, b_1, r_1, k_1, λ_1); there are v treatments arranged in b_1 blocks. In a balanced bipartite block design D_2, with the parameters (v, b_2, r_2, k_{12}, k_{22}, λ_{12}, λ_{22}); there are v treatments arranged in b_2 blocks such that each block containing k distinct treatments is divided into two subblocks containing k_{12} and k_{22} treatments, respectively, where $k_2 = k_{12} + k_{22}$. Each treatment appears in r_2 blocks. Every pair of treatments with different subblocks appears together in λ_{12} blocks and every pair of treatments with the same subblock appears together in λ_{22} blocks. Also in a ternary balanced block design D_3, there are b_3 blocks, each of size k_3, chosen from a set of size v in such a way that each of the v elements occurs r_3 times all together and 0,1 or 2 times in each block and each of the $\binom{v}{2}$ distinct pairs of elements occurs λ_3 times. Any ternary balanced block design is regular, that is, each element occurs singly in ρ_{13} blocks and is repeated in ρ_{23} blocks.

Under the present method of construction, the design *D** yields the parameters $v^* = v$, $b^* = b_1 + 2\,b_2 + b_3$, $r^* = r_1 + 2\,r_2 + r_3$, which are obvious. Since in the original BIB design, balanced bipartite block design and ternary balanced block design; any $(\theta, \emptyset)$ pair occurs in λ_1, λ_2, λ_3 blocks respectively. Thus in a new construction method, the frequency of $(\theta, \emptyset)$ pair in the new design can be calculated as

$$\lambda^* = \lambda_1 + 2(\lambda_{12} + \lambda_{22}) + \lambda_3$$

The calculations on efficiency and variance can be done as follows.

$$N^*\left(N^*\right)' = \left(r_1 + 2r_2 + r_3 + 2\rho_{23}\right) I_v + \left(\lambda_1 + 2(\lambda_{12} + \lambda_{22}) + \lambda_3\right)\left(J_{vv} - I_v\right)$$

$$\left(r^*\right)^{-\delta} N^*\left(k^*\right)^{-\delta}\left(N^*\right)' = \left(\frac{r_1}{k_1} + \frac{2r_2}{k_2} + \frac{r_3 + 2\rho_{23}}{k_3}\right)\frac{1}{(r_1 + 2r_2 + r_3)} I_v + \left(\frac{\lambda_1}{k_1} + \frac{2\lambda_2}{k_2} + \frac{\lambda_3}{k_3}\right)\frac{1}{(r_1 + 2r_2 + r_3)}\left(J_{vv} - I_v\right)$$

which yields

$$\psi^* = 1 - \frac{v}{(r_1 + 2r_2 + r_3)}\left(\frac{\lambda_1}{k_1} + \frac{2\lambda_2}{k_2} + \frac{\lambda_3}{k_3}\right)$$

On the other hand, it holds that

$$\left(r^*\right)^{-\delta} - N^*\left(k^*\right)^{-\delta}\left(N^*\right)' = \left[\left(r_1 + 2r_2 + r_3\right) - \left(\frac{r_1}{k_1} + \frac{2r_2}{k_2} + \frac{r_3 + 2\rho_{23}}{k_3}\right)\right] I_v + \left(\frac{\lambda_1}{k_1} + \frac{2\lambda_2}{k_2} + \frac{\lambda_3}{k_3}\right)\left(J_{vv} - I_v\right)$$

which yields

$$\mu^* = v\left(\frac{\lambda_1}{k_1} + \frac{2\lambda_2}{k_2} + \frac{\lambda_3}{k_3}\right)$$

This completes the proof.

Example 6.2. Consider a Balanced Incomplete Block Design with parameters $v = 5$, $b_1 = 5$, $r_1 = 4$, $k_1 = 4$, $\lambda_1 = 3$; Balanced Bipartite design with parameters BIB design with parameters $v = 5$, $b_2 = 5$, $r_2 = 4$, $k_{12} = 2$, $k_{22} = 2$, $\lambda_{12}=2$, $\lambda_{22} = 1$ and Balanced Ternary Design with parameters $v = 5$, $b_3 = 15$, $r_3 = 9$, $k_3 = 3$, $\lambda_3 = 4$, $\rho_{13} = 7$, $\rho_{23}=1$.

Theorem 6.1 yields a design matrix X of pairwise VB and EB design with parameters

$v^* = 5,\ b^* = 30,\ r^* = 21,\ k^* = [4.1_5'\quad 4.1_5'\quad 4.1_5'\quad 3.1_5'],\ \lambda^* = 13,$

$\mu^* = 17.9167$ and $\psi^* = 0.1468$.

The nested structure is as follows

[(1, 2, 3, 4) (1, 2, 3, 5) (1, 2, 4, 5) (1, 3, 4, 5) (2, 3, 4, 5)]; [(1, 2, 3, 4) (1, 2, 3, 5) (1, 2, 4, 5) (1, 3, 4, 5) (2, 3, 4, 5)]; [(1, 2, 3, 4) (1, 2, 3, 5) (1, 2, 4, 5) (1, 3, 4, 5) (2, 3, 4, 5)]; [(1, 1, 5) (1, 2, 2) (2, 3, 3) (3, 4, 4) (4, 5, 5) (1, 2, 4) (1, 2, 4) (1, 3, 4) (1, 3, 5) (1, 3, 5) (1, 3, 4) (2, 3, 5) (2, 3, 5) (2, 4, 5) (2, 4, 5)].

7. DISCUSSION

The following Tables 1 provides the list of pairwise variance and efficiency balanced block designs for the structure given in (18), which can be obtained by using certain known balanced incomplete, balanced bipartite and ternary balanced block designs.

Table 1

S. No.	v*	b*	r*	k_1^*	k_2^*	k_3^*	λ*	μ*	ψ*	1/μ*	1-ψ*	Reference No.
1	5	25	22	4	5	4	18	20	0.09091	0.05	0.90909	R(4), [3]
2	5	21	20	4	5	5	18	18.75	0.0625	0.05333	0.9375	R(4), [3]
3	5	30	26	4	5	4	21	23.75	0.08654	0.04211	0.91346	R(4), [3]
4	5	20	19	4	5	5	17	17.75	0.06579	0.05634	0.93421	R(4), [3]
5	5	35	24	4	4	3	14	19.5833	0.18403	0.05106	0.81597	R(4), [3]
6	5	40	25	4	3	3	13	20.4167	0.18333	0.04898	0.81667	R(4), [3]
7	5	30	21	4	4	3	13	17.9167	0.14683	0.05581	0.85317	R(4), [3]
8	6	29	28	5	6	6	25	25.8	0.07857	0.03876	0.92143	R(8), [3]
9	11	44	28	5	6	11	18	25.4	0.09286	0.03937	0.90714	R(29), MH(5), [3]
10	11	44	29	6	6	11	19	26.5	0.08621	0.03774	0.91379	R(30), [3]

**The symbols R(α) and MH(α) denote the reference number α in Raghavrao [20] and Marshal Halls [25] list.

8. CONCLUSION

It is well known that pairwise balanced designs are not always efficiency as well as variance balanced. But in this research we have significantly shown that the proposed pairwise balanced designs are efficiency as well as variance balanced. Further there is a scope to propose different methods of construction to obtain the optimum chemical balance weighing designs and pairwise variance and efficiency balanced block designs, which will fulfill the optimality criteria by means of efficiency. In this research paper we also gave the conditions under which the constructed chemical balance weighing designs lead to A-optimal designs. The only limitation of this research is that the obtained pairwise balanced designs are all have large number of replications.

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AUTHORS' BIOGRAPHY

Dr. Shakti Banerjee, He is Professor and Head in School of Statistics, Devi Ahilya Vishwavidyalaya, Indore. His area of specialization is Design of Experiments. He has 25 years of teaching and research experience. His areas of interests are model building, multivariate analysis, operations research, statistical process and quality control. He has produced five PhDs to his credit and currently seven research students are working under his supervision. He has published more than 50 research articles in various esteemed national and international journals. He is member of various professional bodies.

Rashmi Awad, She is PhD student at Devi Ahilya Vishwavidyalaya, Indore. Her area of research is Design of Experiments. She has published seven research articles in leading peer-reviewed journals. She has 10 years experience of teaching in Govt. Holkar Science College, Indore.