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2(1)-PRIME SUBSEMIMODULES OF PARTIAL SEMIMODULES OVER PARTIAL SEMIRINGS

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Abstract: *A partial semiring is a structure possessing an infinitary partial addition and a binary multiplication, subject to a set of axioms. The partial functions under disjoint-domain sums and functional composition is a partial semiring. In this paper we introduce the notions of 2(1)-prime subsemimodule and 2(1)-closed subset in partial semimodules over partial semirings and we obtain some characteristics of 2(1)-prime subsemimodules.*

Keywords: *Multiplication Partial Semimodule, 2(1)-Prime partial ideal, 2(1)-Prime subsemimodule and 2(1)-closed subset.*

1. INTRODUCTION

Partially defined infinitary operations occur in the contexts ranging from integration theory to programming language semantics. The general cardinal algebras studied by Tarski in 1949, Housdorff topological commutative groups studied by Bourbaki in 1966, Σ-structures studied by Higgs in 1980, sum ordered partial monoids & sum ordered partial semirings studied by Arbib, Manes, Benson[2], [4] and Streenstrup[6] are some of the algebraic structures of the above type.

P. Nanda Kumar[5] in 2010 introduced the notion of *1-(2-)* prime ideals in semirings. we extended this notation to partial ideals in partial semirings and we obtained the characteristics of *2(1)*-prime partial ideals[7]. In this paper we introduce the notions of *2(1)*-prime subsemimodules in partial semimodules over partial semirings and characterize *2(1)*-prime subsemimodules interms of *2(1)*-prime partial ideals. We also introduce the notions of *2(1)*-closed subset and obtain the characteristics of *2(1)*-prime subsemimodules in multiplication partial semimodules.

2. PRELIMINARIES

In this section we collect some important definitions and results for our use in this paper.

2.1. Definition. [5] A *partial monoid* is a pair (M, Σ) where M is a nonempty set and Σ is a partial addition defined on some, but not necessarily all families $(x_i : i \in I)$ in M subject to the following axioms:

(i) Unary Sum Axiom. If $(x_i : i \in I)$ is a one element family in M and $I = \{ j \}$, then

$\Sigma(x_i : i \in I)$ is defined and equals x_j.

(ii) Partition-Associativity Axiom. If $(x_i : i \in I)$ is a family in M and $(I_j : j \in J)$ is a partition of

I, then $(x_i : i \in I)$ is summable if and only if $(x_i : i \in I_j)$ is summable for every j in J,

$(\Sigma(x_i : i \in I_j) : j \in J)$ is summable, and $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j):j\in J)$.

2.2. Definition. [4] A *partial semiring* is a quadruple $(R,\Sigma,.,1)$, Where (R,Σ) is a partial monoid with partial addition $\sum$, $(R,.,1)$ is a monoid with multiplicative operation $\cdot$ and unit 1, additive and multiplicative structures obey the following distributive laws: If $\sum(x_i : i \in I)$ is defined in R, then for all y in R, $\sum(y\cdot x_i : i\in I)$ and $\sum(x_i\cdot y : i\in I)$ are defined and $y\cdot[\sum_i x_i]=\sum_i[y\cdot x_i]$, $[\sum_i x_i]\cdot y=\sum_i[x_i\cdot y]$.

2.3. Definition. [1] Let R be a partial semiring. Then a nonempty subset A of R is said to be a *left (right) partial ideal* of R if it satisfies the following:

(i) $(x_i : i \in I)$ is a summable family in R and $x_i \in A$ $\forall\, i \in I$ implies $\Sigma_i x_i \in A$,

(ii) for all $r \in R$ and $x \in A$, $rx \in A$ $(xr \in A)$.

If A is both left and right partial ideal of a partial semiring R, then A is called a *partial ideal* of R.

2.4. Definition. [8] A nonempty subset A of a partial semiring R is said to be *subtractive* if for any a, $b \in R$, $a + b \in A$ and $a \in A$ implies that $b \in A$.

2.5. Definition. [8] Let A be a nonempty subset of a partial semiring R. Then the intersection of all subtractive partial ideals of R containing A is called *subtractive closure of A*. It is denoted by $\overline{A}$.

2.6. Definition. [7] A proper ideal P of a partial semiring R is said to be *2-prime* if and only if for any subtractive partial ideals A, B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

2.7. Definition. [7] A proper ideal P of a partial semiring R is said to be *1-prime* if and only if for any subtractive partial ideal A and a partial ideal B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

2.8. Theorem. [7]] Let R be a complete partial semiring and P be a proper partial ideal of R. Then the following conditions are equivalent:

(i) P is a *2(1)*-prime partial ideal of R

(ii) for any a, $b \in R$, $\overline{<a>}\,\overline{<b>} \subseteq P(\overline{<a>}<b>\subseteq P)$ implies $a \in P$ or $b \in P$.

2.9. Definition. [8] Let $(R,\Sigma,\cdot,1)$ be a partial semiring and $(M,\overline{\Sigma})$ be a partial monoid. Then M is said to be a left partial semimodule over R if there exists a function $*: R\times M \to M : (r,x)\mapsto r*x$ which satisfies the following axioms for x, $(x_i : i\in I)$ in M and r_1, r_2, $(r_j : j\in J)$ in R

(i) if $\overline{\Sigma}_i x_i$ exits then $r*(\overline{\Sigma}_i x_i)=\overline{\Sigma}_i(r*x_i)$,

(ii) if $\sum_j r_j$ exits then $(\sum_j r_j)*x = \overline{\Sigma}_j(r_j*x)$,

(iii) $r_1*(r_2*x)=(r_1\cdot r_2)*x$,

(iv) $1_R*x=x$,

(v) $0_R*x=0_M$.

2.10. Definition. [8] Let $(M,\overline{\Sigma})$ be a left partial semimodule over a partial semiring R. Then a nonempty subset N of M is said to be a *subsemimodule* of M if and only if N is closed under $\overline{\Sigma}$ and $*$.

2.11. Definition. [8] Let N be a subsemimodule of a left partial semimodule M over R.Then $(N:M)=\bigcap\{(N:m)/m\in M\}=\{r\in R/rM\subseteq N\}$ is called the *associated partial ideal* of N.

2.12. Remark. [8] If N is a subtractive subsemimodule of M then its associated partial ideal

$(N:M)$ is a subtractive partial ideal of R.

2.13. Definition. [8] Let M be a partial semimodule over R .Then M is said to *be multiplication partial semimodule* if for any subsemimodule N of M there exists a partial ideal I of R such that $N = I M$.

2.14. Theorem. [8] A partial semimodule M over R is a multiplication partial semimodule if and only if there exists a partial ideal I of R such that $R\,m = I\,M$ for each $m\in M$.

2.15. Definition. [8] Let *M* be a multiplication partial semimodule over *R* and *N*, *K* be subsemimodules of *M* such that *N = I M* and *K = J M* for some partial ideals *I, J* of *R*. Then the *multiplication of N and K* is defined as *NK = (IM)(IJ) = (IJ)M.*

2.16. Definition. [8] Let M be a multiplication partial semimodule over R and m_1, m_2 in M such that R $m_1 = I_1M$ and R $m_2 = I_2M$ for some partial ideals I_1, I_2 of R. Then the multiplication of m_1 and m_2 is defined as $m_1m_2 = (I_1M)(I_2M) = (I_1I_2)M$.

2.17. Definition. [8] Let *M* be a partial semimodule over *R* and *N* be a proper subsemimodule of *M*. Then *N* is said to be a *prime subsemimodule* of *M* if for any $r \in R$ and $n \in M$, $r * n \in N$ implies $r \in (N : M)$ or $n \in N$.

2.18. Theorem. [8] Let *M* be a multiplication partial semimodule over *R* and *N* be a subsemimodule of *M*. Then *N* is prime subsemimodule of *M* if and only if $(N : M)$ is a prime partial ideal of *R*.

2.19. Definition. [8] Let *M* be a multiplication partial semimodule over *R*. A subset *S* of *M* is said to be *multiplication closed subset* (in short *closed subset*) if for any $m, n \in S, mn \bigcap S \neq \Phi$.

3. *2(1)*-PRIME SUBSEMIMODULES

3.1. Definition. Let *R* be a partial semiring, *M* be a partial semimodule over *R* and *N* be a proper subsemimodule of *M*. Then *N* is said to be a *2(1)-prime subsemimodule* of *M* if and only if its associated partial ideal *(N : M)* is a *2(1)*-prime partial ideal of *R*.

3.2. Theorem. Let *M* be a multiplication partial semimodule over *R* and *K* be a subtractive subsemimodule of *M*. Then *K* is prime subsemimodule if and only if *K* is *2(1)*-prime subsemimodule of *M*.

Proof. Suppose *K* is a prime subsemimodule of M. Then *(K : M)* is a prime partial ideal of *R* (By Theorem 3.3.3 of **[8]**). This implies *(K : M)* is a *2(1)*-prime partial ideal of *R* (By Remark 3.3 of **[7]**). Hence *K* is a *2(1)*-prime subsemimodule of *M.*

Conversely suppose that *K* is a *2(1)*-prime subsemimodule of *M*. Then *(K : M)* is a *2(1)*-prime partial ideal of *R*. Since *K* is subtractive subsemimodule of *M*, *(K : M)* is a subtractive partial ideal of *R* (By Remark 2.12). By Theorem 3.9 of **[7]**, *(K : M)* is a prime partial ideal of *R*.

Since *M* is multiplication partial semimodule, by Theorem 3.3.7 of **[8]** *K* is a prime subsemimodule of *M*.

Following is an example of a partial semimodule *M* in which a *2(1)*-prime subsemimodule is not prime subsemimodule.

3.3. Example. Let *R = {0, 1, 2, 3}*. Define *Σ* on *R* as

$$\sum_i x_i = \begin{cases} x_j, if \quad x_i = 0 \forall i \neq j \; for \quad some \quad j \\ 2, if \quad x_j = x_k = 1 \; for \quad some \; j, k \quad and \quad x_i = 0 \forall i \neq j, k \\ 3, if \quad J = \{ i \mid x_i \neq 0 \} is \quad finite \quad and \quad \Sigma(x_i : i \in J) \geq 3 \\ undefined, \qquad otherwise \end{cases}$$

and $\cdot$ is defined as follows:

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	3	3
3	0	3	3	3

Then $(R, \Sigma, \cdot)$ is a partial semiring. Take $M := R$ and $K = \{0, 3\}$. Then M is a partial semimodule over R and K is a subsemimodule of M. Now $(K : M) = \{0, 3\}$ is a *2(1)*-prime partial ideal of R and so, K is a *2(1)*-prime subsemimodule of M. Since $\{0,2,3\}\cdot\{0,2,3\}=\{0,3\}$ and $\{0,2,3\} \not\subset \{0,3\}$, $(K : M)$ is not a prime partial ideal of R. Therefore K is not a prime subsemimodule of M.

3.4. Theorem. Let M be a multiplication partial semimodule over R and K be a proper subsemimodule of M. Then the following conditions are equivalent:

(i) K is a *2*-prime subsemimodule of M,

(ii) for any subtractive subsemimodules U, V of M, $UV \subseteq K$ implies $U \subseteq K$ *or* $V \subseteq K$,

(iii) for any m, n of M, $(\overline{Rm})(\overline{Rn}) \subseteq K$ implies $m \in K$ or $n \in K$.

Proof. **(i)$\Rightarrow$(ii):** Suppose K is a *2*-prime subsemimodule of M. Then $(K : M)$ is a *2*-prime partial ideal of R. Let U, V be subtractive subsemimodules of M such that $UV \subseteq K$. Since M is multiplication partial semimodule, there exists partial ideals I, J of R such that $U = IM$ and $V = JM$. $\Rightarrow (U : M) = I$ and $(V : M) = J$. Since U, V are subtractive subsemimodules of M, $(U : M) = I$ and $(V : M) = J$ are subtractive partial ideals of R. Now $UV \subseteq K \Rightarrow (IM)(JM) \subseteq K \Rightarrow (IJ)M \subseteq K$ $\Rightarrow IJ \subseteq (K : M)$. Since $(K : M)$ is a *2*-prime partial ideal of R, $IJ \subseteq (K : M)$ $I \subseteq (K : M)$ or $J \subseteq (K : M)$. $\Rightarrow IM \subseteq K$ or $JM \subseteq K$ and hence $U \subseteq K$ or $V \subseteq K$.

(ii) $\Rightarrow$(iii): Suppose for any subtractive subsemimodules U, V of M, $UV \subseteq K$ implies $U \subseteq K$ *or* $V \subseteq K$. Let $m, n \in M$ be such that $(\overline{Rm})(\overline{Rn}) \subseteq K$. Since $(\overline{Rm}), (\overline{Rn})$ are subtractive subsemimodules of M, $(\overline{Rm}) \subseteq K$ or $(\overline{Rn}) \subseteq K$ and hence $m \in K$ or $n \in K$.

(iii) $\Rightarrow$(i): Suppose for any m, n of M, $(\overline{Rm})(\overline{Rn}) \subseteq K$ implies $m \in K$ or $n \in K$. First we prove that $(K : M)$ is a *2*-prime partial ideal of R. Let A, B be subtractive partial ideals of R such that $AB \subseteq (K : M)$. Then $(AB)M \subseteq K$. Suppose $A \not\subset (K : M)$ and $B \not\subset (K : M)$. Then $AM \not\subset K$ and $BM \not\subset K$. $\Rightarrow$ $\exists a \in A, b \in B, m, n \in M \ni a * m \in AM, b * n \in BM$, $a * m \notin K \,\&\, b * n \notin K$. $\Rightarrow$ $\overline{R(a*m)} \subseteq AM$ and $\overline{R(b*n)} \subseteq BM$. Now

$(\overline{R(a*m)})(\overline{R(b*n)}) \subseteq (AM)(BM) = (AB)M \subseteq K$. $\Rightarrow a*m \in K$ or $b*n \in K$, a contradiction. Hence $A \subseteq (K : M)$ and $B \subseteq (K : M)$. Therefore $(K : M)$ is a *2*-prime partial ideal of R. Hence K is a *2*-prime subsemimodule of M.

In the similar way we can prove the following theorem.

3.5. Theorem. Let M be a multiplication partial semimodule over R and K be a proper subsemimodule of M. Then the following conditions are equivalent:

(i) K is a *1*-prime subsemimodule of M,

(ii) for any subsemimodules U, V of M, $\overline{U}V \subseteq K$ implies $U \subseteq K$ *or* $V \subseteq K$,

(iii) for any m, n of M, $(\overline{Rm})(Rn) \subseteq K$ implies $m \in K$ or $n \in K$.

4. *2(1)*-Closed Subset of M

4.1. Definition. Let M be a multiplication partial semimodule over R. A subset S of M is said to be a *2(1)-closed subset* of M if for any $m, n \in S, \exists m_1 \in \overline{Rm}, n_1 \in \overline{Rn}(n_1 \in Rn) \ni m_1 n_1 \bigcap S \neq \Phi$.

Clearly every closed subset of M is a *2(1)*-closed subset of M. Following is an example of a partial semimodule M in which a *2(1)*-closed subset is not a closed subset of M.

4.2. Example. Consider the partial semimodule M over R as in the Example 3.3. Clearly M is a multiplication partial semimodule over R. Take $S = \{1, 2\}$. Then S is a *2(1)*-closed subset of M. For $2 \in S, 2 \cdot 2 = 4 \notin S$. Hence S is not a closed subset of M.

4.3. Theorem. Let *M* be a multiplication partial semimodule over *R* and *K* be a proper subsemimodule of *M*. Then *K* is a *2(1)*-prime subsemimodule of *M* if and only if *M* \ *K* is a

2(1)-closed subset of *M*.

Proof. Suppose *K* is a *2(1)*-prime subsemimodule of *M*. Let $m, n \in M \setminus K$. Then $m, n \notin K$.

By the Theorem 3.4, $(\overline{Rm})(\overline{Rn}) \not\subset K$. $\Rightarrow$ $\exists x \in (\overline{Rm})(\overline{Rn}) \ni x \notin K$. Since $x \in (\overline{Rm})(\overline{Rn})$, $x = \Sigma_i m_i n_i$ for some $m_i \in \overline{Rm}$ and $n_i \in \overline{Rn}$. Since $x \notin K$, $\Sigma_i m_i n_i \notin K$. $\Rightarrow$

$\exists m_i' \in \overline{Rm} \,\&\, n_i' \in \overline{Rn} \ni m_i' n_i' \notin K$. $\Rightarrow$ $m_i' n_i' \cap M \setminus K \neq \Phi$. Hence *M* \ *K* is a

2(1)-closed subset of *M*.

Conversely suppose that *M* \ *K* is a *2(1)*-closed subset of *M*. Let $m, n \in M$ be such that $(\overline{Rm})(\overline{Rn}) \subseteq K$. Suppose if $m \,\&\, n \notin K$. Then $m \,\&\, n \in M \setminus K$. $\Rightarrow$

$\exists m_1' \in \overline{Rm} \,\&\, n_1' \in \overline{Rn} \ni m_1' n_1' \cap M \setminus K \neq \Phi$. $\Rightarrow$ $\exists x \in (\overline{Rm})(\overline{Rn}) \,\&\, x \notin K$. $\Rightarrow$ $(\overline{Rm})(\overline{Rn}) \not\subset K$, a contradiction. Hence *K* is a *2(1)*-prime subsemimodule of *M*.

4.4. Theorem. Let *M* be a multiplication partial semimodule over *R*. Then every *2(1)*-prime subsemimodule of *M* contains a minimal *2(1)*-prime subsemimodule of *M*.

Proof. Take $\mathcal{T}$ = {*P* | *P* is a *2(1)*-prime subsemimodule of *M* such that $P \subseteq K$ }. Clearly $K \in \mathcal{T}$. Hence $(\mathcal{T}, \subseteq)$ is a nonempty partial ordered set. Let $\{ H_j \mid j \in J \}$ be a descending chain of elements in $\mathcal{T}$. Put $H = \bigcup_{j \in J} H_j$. Then *H* is clearly a subsemimodule of *M* containing *K*. Now we prove that *H* is a *2*-prime subsemimodule of *M*: Let $a, b \in M$ such that $(\overline{Ra})(\overline{Rb}) \subseteq H$ and suppose $a \notin H$. Then $a \notin H_k$ for some $k \in J$. Case(i): If $j \leq k$. Since $(\overline{Ra})(\overline{Rb}) \subseteq H_k$ and $a \notin H_k$, we have $b \in H_k$. Since $j \leq k$, $H_k \subseteq H_j$ and hence $b \in H_j$. Case(ii): If $j > k$. Then $H_j \subseteq H_k$. Since $a \notin H_k$, $a \notin H_j$. Now $(\overline{Ra})(\overline{Rb}) \subseteq H_j$ and $a \notin H_j$, we have $b \in H_j$. Hence $b \in H_j$ for all $j \in J$. $\Rightarrow b \in H$. Hence *H* is a *2*-prime subsemimodule of *M* containing *K*. Therefore *H* is a lower bound of $\{ H_j \mid j \in J \}$ in $\mathcal{T}$. By Zorn's lemma, $\mathcal{T}$ has a minimal element. Hence the theorem.

5. Conclusion

In this paper we introduce the notion of *2(1)*-prime subsemimodule in partial semimodule and obtained the characterization of *2(1)*-prime subsemimodules interms of *2(1)*-prime partial ideals. Also we introduced the notion of *2(1)*-closed subset and obtained the characteristics of *2(1)*-prime subsemimodule in a multiplication partial semimodule.

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