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Comparison Results for Nonlinear Difference Equations

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Abstract: *The existence of solution and extremal solutions of equation are obtained.*

$$\Delta u(t) = g(t, u(t)); \quad u(t_o) = u_o$$

Some comparison theorems are proved and are extended to finite system of difference inequalities. To extend the results, mixed monotone property is used.

Keywords: *Fixed point Theorems, Mixed monotone operators, Nonlinear operators, Difference inequalities*

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1. Introduction

In recent years, the theory and applications of difference equations are found to be more useful in the engineering field. Agarwal [1], Kelley and Peterson [3] had developed the theory of difference equations and difference inequalities. Some comparison theorems are obtained by Eloe [2].

In the present paper, the existence of solution of equation

$$\Delta u(t) = g(t, u(t)); \quad u(t_o) = u_o \tag{1.1}$$

is proved via Schauder's fixed point theorem. Some comparison theorems are obtained in section 2 and the existence of extremal solutions of equation (1.1) are obtained in section 3. The mixed monotone property, as defined in [4,5] is used to extend comparison theorems to finite system of difference inequalities.

2. Existence of Solution

Let $J_0 = \{t_{0,} t_0 + 1, \dots \dots \dots, t_0 + a\},\ t_0 \in R$ and E be an open subset of R. Consider the difference equation

$$\Delta u(t) = g(t, u(t)); \quad u(t_0) = u_0 \tag{2.1}$$

where $u_0 \in E$; $t \in J$, $u : J \to R$ with $u(t) \in E$ and $g : J \times E \to R$.

A function $\phi : J \to R$ is said to be solution of I.V.P. (2.1) if it satisfies

$$\Delta \phi(t) = g(t, \phi(t)) \; ; \quad \phi(t_0) = u_0 .$$

The I.V.P. (2.1) is equivalent to the problem

$$u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)) \tag{2.2}$$

By summation convention, $\sum_{s=t_0}^{t_0-1} g(s, u(s)) = 0$ and so the equation (2.2) satisfies equation (2.1).

Let B= *{real valued functions defined on J.}* and define $T : B \to B$ by

$$Tu(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)) \tag{2.3}$$

The solutions of equation (2.1) are necessarily fixed points of the operator *T*. Since the steps of analysis are reversible, it is also true that all fixed points of *T* are solutions of (2.1).

Since the initial value problem for difference equations often have multiple solutions, it is useful to have a result that yields solutions without the implications that the solutions must be unique. The existence theorem of this type to be presented in this section will be based on the following version of Schauder's theorem.

Theorem 2.1: *(Schauder's fixed point theorem) Let M be a nonempty, closed, bounded, convex subset of a Banach space X and suppose* $T : M \to M$ *is a compact operator. Then T has a fixed point.*

Now we prove our basic existence theorem.

Theorem 2.2: *Let* $g : R_0 \to R$, *where* $R_0 = \{(t, u) \in J \times E \text{ with } |u - u_0| \le b\}$*;* $|g(t,u)| \le M$ *on* R_0 *and g(t,u) is continuous in u. Then the I.V.P. (2.1) has a solution on* $[t_0, t_0 + \alpha]$ *where* $\alpha = \min\left\{a, \frac{b}{M}\right\}$.

Proof: Let *B={real valued functions defined on* $[t_0, t_0 + \alpha]$*}*. Define a norm $\|.\|$ on *B* by $\|x\| =$ *sup* $\{|x(t)| : t_0 \le t \le t_0 + \alpha\}$ for $x \in B$, then *B* is a Banach space. Assume $B_0 = \{u \in B : \|u - u_0\| \le b\}$. So B_0 is closed and convex subset of *B*.

Now define a mapping *T* on B_0 by

$$Tu(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).$$

Clearly *T* is continuous. For $u \in B_0$ we have

$$|Tu(t) - u_0| \le \sum_{s=t_0}^{t-1} |g(s, u(s))| \le \sum_{s=t_0}^{t-1} M \le M\alpha \le b.$$

$$\therefore \|Tu - u_0\| = \sup\{Tu(t) - u_0| : t_0 \le t \le t_0 + \alpha\} \le b.$$

$\therefore Tu$ is in B_0. Since B_0 is bounded and *T* is continuous, *T* is also completely continuous. Hence conclusion follows from Theorem (2.1).

3. Comparison Theorems

Definition 3.1: A real valued function $\alpha(t)$ on *J* is said to be lower solution for (2.1) if

$$\Delta\alpha(t) \le g(t, \alpha(t)); \quad \alpha(t_0) \le u_0 \tag{3.1}$$

Similarly $\beta(t)$ is said to be an upper solution for (2.1) if

$$\Delta\beta(t) \ge g(t, \beta(t)); \quad \beta(t_0) \ge u_0 \quad \text{for } t \in J \tag{3.2}$$

Theorem 3.1: *Assume*

(H₁) → *(H_1) g(t, u) is nondecreasing in u for* $t \in J$ *and* $u \in E$.

(H_2) $\alpha(t)$ *and* $\beta(t)$ *are lower and upper solutions of equation (2.1).*

(H_3) $\alpha(t_0) < \beta(t_0)$ *(3.3)*

Then

$\alpha(t) < \beta(t)$ *for* $t \in J$. *(3.4)*

Proof: If the assertion (3.4) is not true then the set $Z = \{t \in J : \beta(t) \le \alpha(t)\}$ is nonempty.

Let $t_1 = \inf Z$. Clearly $t_0 < t_1$,

$$\beta(t_1) \le \alpha(t_1) \tag{3.5}$$

and $$\beta(t_1 - 1) > \alpha(t_1 - 1) \tag{3.6}$$

Inequalities (3.5) and (3.6) give

$$\Delta \beta(t_1 - 1) < \Delta \alpha(t_1 - 1)$$

i.e. $$g(t_1 - 1, \beta(t_1 - 1)) < g((t_1 - 1, \alpha(t_1 - 1)) \tag{3.7}$$

The inequalities (3.6) and (3.7) contradict to *(H_1)*. So *Z* is empty and assertion (3.4) follows.

Remark 3.1: If $\alpha(t_0) \le \beta(t_0)$ and any one of the inequalities,

$$\Delta \alpha(t) \le g(t, \alpha(t)) \text{ and } \Delta \beta(t) \ge g(t, \beta(t)) \tag{3.8}$$

is strict then (3.4) holds for $t \in (t_0, t_0 + a]$.

Corollary 3.1: *Let* $g_1, g_2 : J \times E \to R$ *and* $g_1(t,u) < g_2(t,u)$ *for* $(t,u) \in J \times E$. *Suppose* $g_1(t,u)$ *is nondecreasing in u and* $g_2(t,u)$ *is nonincreasing in u for* $t \in J$. *Let $u_i(t)$ be solution of equation* $\Delta u_i(t) = g_i(t, u_i(t))$; $i = 1,2$ *existing on J such that* $u_1(t_0) < u_2(t_0)$ *then* $u_1(t) < u_2(t)$ *for* $t \in J$.

Proof: Let $u(t) = u_2(t) - u_1(t)$, $g(t,u) = g_2(t,u) - g_1(t,u)$.

It is obvious that $g(t,u)$ is nondecreasing in *u* and $u(t_0) > 0$.

Consider the difference equation

$$\Delta u(t) = g(t, u(t)); \quad u(t_0) = u_2(t_0) - u_1(t_0) \tag{3.9}$$

Clearly the zero function; $v(t)=0$ for all $t \in J$ is a lower solution of (3.9) and $v(t_0) < u(t_0)$. By Theorem 3.1, we get $v(t) < u(t)$; $t \in J$ .Hence $u_1(t) < u_2(t)$ for $t \in J$.

Theorem 3.2: *Assume (H_1), (H_2) and*

(H_4) $\alpha(t_0) = u_0 = \beta(t_0)$

(H_5) $\alpha(t) < g(t, \alpha(t))$; *(3.10)*

$\beta(t) > g(t, \beta(t))$ *(3.11)*

If u(t) is any solution of (2.1) then

$\alpha(t) < u(t) < \beta(t)$ *for* $t \in J \setminus \{t_0\}$ *(3.12)*

Proof: Since $u(t)$ is any solution of (2.1) we can write

$$\Delta u(t) \le g(t, u(t)), \quad u(t_0) \le u_0 \tag{3.13}$$

From (3.13) and remark 3.1, we have $u(t) < \beta(t)$ for $t \in J \setminus \{t_0\}$. Similarly the left part of inequality (3.12) follows.

Theorem 3.3: Assume (H_1), (H_2) and $\alpha(t_0) < u_0 < \beta(t_0)$. If $u(t)$ is any solution of (2.1) then (3.12) holds for $t \in J$.

The proof of this Theorem follows from Theorem 3.1.

4. Maximal and Minimal Solutions

The notion of maximal and minimal solutions of equation (2.1) will now be introduced.

Definition 4.1: Let $r(t)$ be any solution of equation (2.1) on J. Then $r(t)$ is said to be maximal solution of (2.1) if for every solution $u(t)$ of (2.1) existing on J, the inequality $u(t) \le r(t)$ holds for $t \in J$.

A solution $\rho(t)$ of (2.1) is said to be minimal solution of (2.1) if $\rho(t) \le u(t)$ for $t \in J$.

Now we prove the existence of extremal solutions of equation (2.1)

Theorem 4.1: *In the hypothesis of Theorem 2.2 assume g(t,u) is non- decreasing in u for all* $t \in J$ *.Then the equation (2.1) has maximal and minimal solutions on* $[t_0, t_0 + \alpha]$ *where* $\alpha = \min\left\{a, \frac{b}{2M+b}\right\}$.

Proof: We shall prove the existence of maximal solution only, the case of minimal solution is similar.

Let $o < \in < \frac{b}{2}$. Consider the difference equation

$$\Delta u(t) = g(t,u) + \in, \quad u(t_0) = u_0 + \in \tag{4.1}$$

Clearly the function $g_{\in}(t,u) = g(t,u) + \in$ is continuous with respect to u, for (t,u) in $R_{\in}$, where $R_{\in} = \left\{(t,u) : t \in J; |u - (u_0 + \in)| < \frac{b}{2}\right\}$.

Obviously $R_{\in} \subset R_0$ and $|g_{\in}(t,u)| \le M + \frac{b}{2}$ on $R_{\in}$.

Theorem 2.2 insure that the I.V.P. (4.1) has a solution $u(t,\in)$ on $[t_0, t_0 + \alpha]$ where $\alpha = \min\left\{a, \frac{b}{2M+b}\right\}$. *For* $0 < \in_2 < \in_1 \le \in$, we have

$$u(t_{0,} \in_2) < u(t_0 + \in_1);$$

$$\Delta u(t,\in_2) \le g(t, u(t,\in_2)) + \in_2,$$

$$\Delta u(t,\in_1) = g(t, u(t,\in_1)) + \in_1 > g(t, u(t,\in_1)) + \in_2$$

we can apply Theorem 3.1 to get $u(t,\in_2) < u(t,\in_1)$ for $t \in [t_0, t_0 + \alpha]$.

The family $\left\{u(t,\in) : t \in [t_0, t_0 + \alpha], 0 < \in < \frac{b}{2}\right\}$ is equicontinuous and uniformly bounded on $[t_0, t_0 + \alpha]$. It follows by Ascoli – Arzela theorem that there exists a decreasing sequence $\{\in_n\}$ such that $\in_n \to 0$ as $n \to \infty$ and the uniform limit $r(t) = \lim_{n \to \infty} u(t, \in_n)$ exists on

$[t_0, t_0 + \alpha]$. The uniform continuity of $g(t,u)$ with respect u implies that $g_{\in}(t, u(t, \in))$ converges to $g(t, r(t))$ .The solution of equation (4.1) for $\in = \in_n$ is given by

$$u(t, \in_n) = u_0 + (t - t_0) \in_n + \sum_{s=t_0}^{t-1} g_{\in_n}(s, u(s, \in_n)).$$

Letting $n \to \infty$, $r(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, r(s))$

$$\Rightarrow \quad \Delta r(t) = g(t, r(t)); \quad r(t_0) = u_0 .$$

Hence $r(t)$ is a solution of (2.1) on $[t_0, t_0 + \alpha]$.We shall prove that $r(t)$ is desired maximal solution of (2.1) on $[t_0, t_0 + \alpha]$.

Let $u(t)$ be any solution of equation (2.1) on $[t_0, t_0 + \alpha]$. We observe that

$$u(t_0) = u_0 < u_0 + \in = u(t_0, \in),$$

$$\Delta u(t) = g(t, u(t)) < g(t, u(t)) + \in .$$

$$\Delta u(t, \in) = g_{\in}(t, u(t, \in)) \geq g(t, u(t, \in)) + \in$$

Theorem 3.1 implies that $u(t) < u(t, \in)$, for t in $[t_0, t_0 + \alpha]$ and $\in < \frac{b}{2}$. Letting $\in \to 0$, we get $u(t) \leq r(t)$ on $[t_0, t_0 + \alpha]$. Therefore $r(t)$ is maximal solution of equation (2.1) on $[t_0, t_0 + \alpha]$. This completes the proof.

Theorem 4.2: *Assume that hypothesis of Theorem 4.1 holds. Let $m : J \to R$ satisfies.*

i) $(t, m(t)) \in R_0$

ii) $m(t_0) \leq u_0$ *(4.2)*

iii) $\Delta m(t) \leq g(t, m(t))$ *(4.3)*

for $t \in [t_0, t_0 + \alpha]$ *where* $\alpha = \min \left\{ a, \frac{b}{2M + b} \right\}$. *If r(t) is maximal solution of (2.1) on* $[t_0, t_0 + \alpha]$ *then* $m(t) \leq r(t)$ *on* $[t_0, t_0 + \alpha]$.

Proof: Consider the difference equation

$$\Delta u(t) = g(t, u(t)) + \in; \quad u(t_0) = u_0 + \in \tag{4.4}$$

where $\in > 0$ is sufficiently small. By Theorem 4.1 there is τ such that the equation (4.4) has maximal solution $r(t, \in)$ on $[t_0, t_0 + \tau]$ and

$$r(t) = \lim_{\in \to \infty} r(t, \in) \tag{4.5}$$

uniformly on $[t_0, t_0 + \tau]$. By (4.2) we have

$$m(t_0) \leq u_0 < u_0 + \in = r(t_0, \in) \tag{4.6}$$

$$\Delta r(t, \in) = g(t, r(t, \in)) + \in > g(t, r(t, \in)) \tag{4.7}$$

From (4.2), (4.6),(4.7) and Theorem 3.1 we conclude that

$$m(t) < r(t, \in) \tag{4.8}$$

on $[t_0, t_0 + \tau]$. The inequality (4.8) together with (4.5) proves the assertion of theorem.

Remark 4.1: In the Theorem 4.2 if the inequalities (4.2), (4.3) are reversed then we have $m(t) \geq \rho(t)$ on $[t_0, t_0 + \alpha]$ where $\rho(t)$ is minimal solution of (2.1).

5. FINITE SYSTEMS OF DIFFERENCE INEQUALITIES

Many of the results considered so far for scalar difference inequalities will now be extended, in this section, to finite systems of difference inequalities. To avoid repetition, let us agree on the following: the subscript i ranges over the integers 1,2, ….. n; let $0 \leq k \leq n$, the subscripts p and q range over the integers 1,2, …. k and k+1, k+2, ….n, respectively. We shall use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let $J = \{t_0, t_0+1, \dots t_0 + a\}$; E an open subset of R^n, Eucliden space of dimension n. Consider the system of difference equations with an initial condition, written in the vectorial form

$$\Delta u(t) = g(t, u(t)); \quad u(t_0) = u_0 \tag{5.1}$$

where $u_0 \in E$; $u : J \to E$; $g : J \times E \to R^n$.

Definition 5.1: A function $\alpha : J \to E$ is said to be k lower ($n - k$) upper solution of equation (5.1) if

$$\Delta \alpha_p(t) \leq g_p(t, \alpha(t)) \tag{5.2}$$

$$\alpha_p(t_0) \leq u_{0,p} \quad ;$$

$$\Delta \alpha_q(t) \geq g_q(t, \alpha(t)) \tag{5.3}$$

$$\alpha_q(t_0) \geq u_{0,q}$$

hold for $t \in J$. A function $\beta(t)$ is said to be k upper ($n – k$) lower solution of (5.1) if

$$\Delta \beta_p(t) \geq g_p(t, \beta(t)) \tag{5.4}$$

$$\beta_p(t_0) \geq u_{0,p}$$

$$\Delta \beta_q(t) \leq g_q(t, \beta(t)) \tag{5.5}$$

$$\beta_q(t_0) \leq u_{0,q}$$

for $t \in J$. These definitions include the definitions of lower and upper solutions as special cases, viz. $k = n$ or $k = 0$.

Definition 5.2: The function $g(t,u)$ is said to possess mixed monotone property if the following conditions hold for $t \in J$

i) $g_p(t,u)$ is nondecreasing in u_j ; $j = 1,2 \dots k$ and nonincreasing in u_q.

ii) $g_q(t,u)$ is nonincreasing in u_p and nondecreasing in u_j for $j = k+1, k+2, \dots n$.

Evidently, the particular cases $k = n$ and $k = 0$ in the mixed monotone property correspond to monotone nondecreasing and nonincreasing property of the function $g(t,u)$ respectively.

Theorem 5.1: *Assume that*

(A_1) $g(t,u)$ possess mixed monotone property,

(A_2) α *and* β *are k lower (n – k) upper and k upper (n – k) lower solutions of equation (5.1) respectively,*

(A_3) $\alpha_p(t_0) < \beta_p(t_0)$; $\alpha_q(t_0) > \beta_q(t_0)$. *(5.6)*

iv) Then

$$\alpha_p(t) < \beta_p(t); \quad \alpha_q(t) > \beta_q(t) \tag{5.7}$$

for $t \in J$.

Proof: Define $m_p(t) = \beta_p(t) - \alpha_p(t)$ and $m_q(t) = \alpha_q(t) - \beta_q(t)$. By (5.6); $m_i(t_0) > 0; \quad i = 1,2,\dots\dots n$. Suppose assertion (5.7) is not true. Then the set $Z = \bigcup_{i=1}^{n}\{t \in J; \quad m_i(t) \le 0\}$ is nonempty. Let $t_1 = \min Z$. It is obvious that $t_1 > t_0$. There exists j such that $m_j(t_1) \le 0$ and $m_i(t_1 - 1) > 0$ for all i . Therefore

$$\beta_p(t_1 - 1) > \alpha_p(t_1 - 1) \tag{5.8}$$

and
$$\beta_q(t_1 - 1) < \alpha_q(t_1 - 1) \ . \tag{5.9}$$

So we have

$$\Delta m_j(t_1 - 1) = m_j(t_1) - m_j(t_1 - 1) < 0 . \tag{5.10}$$

Now assume $i \le j \le k$, we observe that

$$\Delta m_j(t_1 - 1) = \Delta\beta_j(t_1 - 1) - \Delta\alpha_j(t_1 - 1)$$

$$\ge g_j(t_1 - 1, \beta(t_1 - 1)) - g_j(t_1 - 1, \alpha(t_1 - 1)) \ . \tag{5.11}$$

Inequalities (5.8),(5.9),(5.11) and mixed monotone property of *g(t, u)* imply that $\Delta m_j(t_1 - 1) \ge 0$ which contradicts to (5.10).

If it is assumed that $k + 1 \le j \le n$, arguing as before, we can obtain a contradiction to (5.10) . Therefore the set *Z* is empty and hence the assertion (5.7) follows.

Remark 5.1: Assume $\alpha_p(t_0) \le \beta_p(t_0)$; $\alpha_q(t_0) \ge \beta_q(t_0)$. If one of the inequalities (5.2) or (5.4) and one of the inequalities (5.3) or (5.5) are strict then (5.7) hold for $t \in J \setminus \{t_0\}$.

Corollary 5.1: Assume *g(t,u)* is monotone nondecreasing in *u*.Suppose $\alpha, \beta : J \to E$ satisfy $\Delta\alpha(t) \le g(t, \alpha(t))$; $\Delta\beta(t) \ge g(t, \beta(t))$. If $\alpha(t_0) < \beta(t_0)$ then $\alpha(t) < \beta(t)$ for $t \in J$.

Analogous to Theorem 3.2 and Theorem 3.3 we have the following theorems.

Theorem 5.2: *Assume (A_1), (A_2),* $\alpha(t_0) = u_0 = \beta(t_0)$ *and inequalities in (5.2) – (5.4) are strict. If u(t) is any solution of equation (5.1) then*

$$\alpha_p(t) < u_p(t) < \beta_p(t) \tag{5.12}$$

and
$$\alpha_q(t) > u_q(t) > \beta_q(t) \tag{5.13}$$

for $t \in J \setminus \{t_0\}$.

Theorem 5.3: Assume *(A_1), (A_2),* $\alpha_p(t_0) < u_p(t_0) < \beta_p(t_0)$ *and* $\alpha_q(t_0) > u_q(t_0) > \beta_q(t_0)$. *If u(t) is any solution of (5.1) then the assertions (5.12) and (5.13) hold for* $t \in J$.

Remark 5.2: Theorems 5.1, 5.2 and 5.3 are extensions of Theorems 3.1, 3.2 and 3.3 respectively.

6. Conclusion

Applications of Difference equations are found more useful in the engineering field. The existence fixed point theorems tool is applicable to discuss the existence and uniqueness of the solution of difference equations. the comparison results obtain in this work are used to compare the maximal and minimal solutions which is obtain using mixed monotone property.

References

[1]. R. Agarwal, “Difference equations and inequalities: Theory, Methods and Applications”, *Marcel Dekker, New York, (1991).*

[2]. P. Eloe, “A comparison theorem for linear difference equations”, *Proc. Amer. Math. Soc. 103(1988), 451-457.*

[3]. Kelley and Peterson, “Difference equations”, *Academic Press, (2001).*

[4]. Borkar V. C. and Patil S. T. “Class of Mixed monotone operators and its application to operator equation”,*Bull of pure and applied Sciences, Vol 22E(No.1) 2003, 179-187.*

[5]. V.Lakshmikantham and S. Leela, “Differential and integral inequalities, Theory and Applications”, *Vol.1, Academic Press (1969).*

[6]. Eberhard Zeidler,“Nonlinear functional Analysis and its Applications ”, *Springer Verlag (1992).*

[7]. T Suzuki, Fixed point theorem in complete metric space “Nonlinear Analysis and convex Analysis (W.Takahashi Ed) Vol. 939 PP 173-182 (RIMS Kokyuroku 1996)

[8]. V.C. Borkar, Extermal Solution of Boundary Value Problems Using Fixed Point Theorems, IOSR Journal of Engineering, May2012, Vol-2(5)P.1196-1199.

[9]. V.C. Borkar and S.T. Patil, Existence and uniqueness of solutions in order Banach Spaces, Journal of pure and applied Mathematical Sciences , Vol LXI No-1,2(X)March 2005 P.53-59

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