International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 4, April 2015, PP 15-20
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

A New Auxiliary Equation Method and Novel Interaction Solutions of the Fifth-Order KDV Equation with Variable Coefficients

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Abstract: *In this paper, a new auxiliary equation method is presented. Analytical multiple function solutions including trigonometric function, exponential function, elliptic function and other functions can be easily obtained. Novel exact interaction solutions of the fifth-order KdV equation with variable coefficients are obtained successfully by using this new auxiliary equation method. It is very significant to help physicists analyze special phenomena in their relevant fields accurately.*

Keywords: *The fifth-order KdV equation with variable coefficients, new auxiliary equation method , interaction solution.*

1. INTRODUCTION

As we known, the complicated nature phenomena are often well described by nonlinear partial differential equations. These equations with variable coefficients are more realistic than constant-coefficient equations in different physics fields[1]. The most representative nonlinear equation is KdV equation with variable coefficients[2].

$$u_t + \alpha(t)uu_{xxx} + \beta(t)u_x u_{xx} + \gamma(t)u^2 u_x + u_{xxxxx} = 0, \qquad (1)$$

where $\alpha(t), \beta(t), \gamma(t)$ are arbitrary function of t. The Korteweg-de Vries (KdV) equation is derived by Korteweg and de Vries to model the evolution of shallow water wave in 1895. The fifth-order KdV equation with variable coefficients can better describe the phenomena of shallow water wave movements.

Many fifth-order physical models are governed by the fifth-order KdV equation with variable coefficients. Such as:

SK equation:

$$u_t + 5uu_{xxx} + 5u_x u_{xx} + 5u^2 u_x + u_{xxxxx} = 0 \qquad (2)$$

KK equation:

$$u_t + 5uu_{xxx} + \frac{25}{2}u_x u_{xx} + 5u^2 u_x + u_{xxxxx} = 0 \qquad (3),$$

Soliton solutions of these equations have important applications in nonlinear optics, theoretical physics, plasma physics, fluid dynamics, semiconductors and other fields. It is meaningful to solve various exact solutions including trigonometric function, exponential function, elliptic function and other functions. The investigation of such analytical solutions helps us to understand the complicated physics phenomena well. In the past decades, many methods are proposed to obtain exact solutions of nonlinear partial differential equations: such as inverse scattering theory[3], Hirota's bilinear method[4], the truncated Painlevé expansion[5], Darboux transformation[6] and so on. In recent years, a large number of powerful methods to solve nonlinear partial differential equations are considered. One of important methods is the auxiliary equation method, it includes the homogeneous balance method[7], sine-cosine method[8], the sech-function method[9], the hyperbolic tangent function method[10,11], the multiple exp-function method[12,13] , the G'/G-expansion method[14], the generalize G'/G method[15], they all are collectively known as the auxiliary equation method. It has attracted extensive attention as its concise and understandable. Ma[16], Chen[17,18], Chen[19-24] are devoted to constructing special interaction soliton solutions by using combination of auxiliary equations and get great success. But the solutions of the solvable auxiliary equation are singular soliton solutions. In this paper, this novel auxiliary equation can successfully to obtain multiple function solutions which including trigonometric function, exponential function, elliptic function and other functions.

This paper is organized as follow: a new auxiliary equation: $\phi''=a+b\phi+c\phi^3$ which we find out multiple function solutions in section 2. In section 3, we introduce this new auxiliary equation method, it can be applied to many nonlinear partial differential equations in different fields effectively. In section 4, this method is applied to the fifth-order KdV equation with variable coefficients successfully. Many new exact interaction solutions are obtained. Some conclusions and discussions are given in section 5.

2. The New Solutions of the Novel Auxiliary Equation

For the novel auxiliary equation read:

$$\phi''=a+b\phi+c\phi^3 \tag{4}$$

where $\phi''=\phi''(\xi)$. We obtain new multiple solutions of Eq.(4) in the following cases:

case 1: $a=0, b=-2, c=\frac{2}{(a_1A_1)^2}$

$$\phi_1(\xi)=a_1A_1\frac{A_1C_1\tanh(\xi)+C_2\tan(\xi)}{A_1C_1+C_2\tan(\xi)\coth(\xi)}, \tag{5}$$

$$\phi_2(\xi)=a_1A_1\frac{C_1\tan(\xi)+A_1C_2\coth(\xi)}{C_1\tanh(\xi)\tan(\xi)+A_1C_2}, \tag{6}$$

$$\phi_3(\xi)=a_1A_1\frac{A_1C_1\tanh(\xi)+C_2\cot(\xi)}{A_1C_1+C_2\cot(\xi)\coth(\xi)}, \tag{7}$$

$$\phi_4(\xi)=a_1A_1\frac{C_1\cot(\xi)+A_1C_2\coth(\xi)}{C_1\tanh(\xi)\cot(\xi)+A_1C_2}, \tag{8}$$

$$\phi_5(\xi)=a_1A_1\frac{C_1+A_1C_2\tan(\xi)\coth(\xi)}{C_1\tanh(\xi)+A_1C_2\tan(\xi)}, \tag{9}$$

$$\phi_6(\xi)=a_1A_1\frac{A_1C_1\tan(\xi)\tan(\xi)+C_2}{A_1C_1\tan(\xi)+C_2\coth(\xi)}, \tag{10}$$

$$\phi_7(\xi)=a_1A_1\frac{A_1C_1\tanh(\xi)\cot(\xi)+C_2}{A_1C_1\cot(\xi)+C_2\coth(\xi)}, \tag{11}$$

$$\phi_8(\xi)=a_1A_1\frac{C_1+A_1C_2\cot(\xi)\coth(\xi)}{C_1\tanh(\xi)+A_1C_2\cot(\xi)}. \tag{12}$$

case 2: $a=2(a_1-2a_0), b=1, c=0$

$$\phi_9(\xi)=a_0+\frac{a_1\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}{1-\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}+a_3(\exp(\frac{\xi}{2}))^2 \tag{13}$$

$$\phi_{10}(\xi)=a_0+\frac{a_1\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}{1-\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}+a_3(\exp(-\frac{\xi}{2}))^2, \tag{14}$$

where $a_0, a_1, a_3, A_1, C_1, C_2$ are arbitrary constants. Some imaginary solutions are omitted above, therefore real interaction solutions of nonlinear partial differential equations are obtained when we apply this auxiliary equation into equations.

3. The New Auxiliary Equation Method

Step1: For a given nonlinear partial differential equation with independent variables $t, x, y \ldots$:

$$P(t, x, u, u_x, u_y, u_t, u_{xt}, u_{yt}, u_{xy}, u_{xx}, \cdots) = 0 \tag{15}$$

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Step2: We make a transformation as follow:

$$u(x,t) = u(\xi) = u(k(t)x - v(t)), \tag{16}$$

where $k(t), v(t)$ are arbitrary function of t.

Step3: Inserting eq(16) into eq(15), we yield an ordinary differential equation:

$$P(t, x, k', v', u, u', u'' \cdots) = 0, \tag{17}$$

Step4: We assume exact solutions of eq(15) in the following form:

$$u(\xi) = \sum_{i=-m}^{m} b_i(t)\phi^i(\xi), \tag{18}$$

where m is positive integer, it is determined by the balance principle in eq(15). $\phi(\xi)$ satisfies the auxiliary equation eq(4). Substituting eq(18) into eq(17). We obtain a set of algebra equations when set each coefficients of $\phi^j(\xi)\phi'(\xi)$ to zeros. There $b_i(t)(-m \le i \le m), k(t), v(t)$ will be determined by solving a set of algebra equations.

4. Application to the fifth-order KdV Equation with Variable Coefficients

The fifth-order KdV equation with variable coefficients:

$$u_t + \alpha(t)uu_{xxx} + \beta(t)u_x u_{xx} + \gamma(t)u^2 u_x + u_{xxxxx} = 0, \tag{19}$$

Where $\alpha(t), \beta(t), \gamma(t)$ are arbitrary functions of t.

We have the hypothesis in the following terms is obtained:

$$u(\xi) = \sum_{i=-m}^{m} b_i(t)\phi^i(\xi), u(\xi) = u(k(t)x - v(t)), \tag{20}$$

where $b_i(t)(-m \le i \le m), k(t), v(t)$ are functions of t. Where m is positive integer and equate to 2, it is determined by balancing the linear term of u_{xxxxx} and the nonlinear terms of uu_{xxx} or $u_x u_{xx}$. $\phi(\xi)$ satisfies the auxiliary equation: $\phi'' = a + b\phi + c\phi^3$. For the sake of simplicity, we take the exact solutions of eq(19) as following form:

$$u(\xi) = b_0(t) + b_1(t)\phi(\xi) + \frac{b_{-1}(t)}{\phi(\xi)}, u(\xi) = u(k(t)x - v(t)), \tag{21}$$

where $b_0(t), b_1(t), b_{-1}(t), k(t), v(t)$ are arbitrary function of t, they could be all determined in the later. Hence, substituting eq(21) into eq(19) along with aid of the auxiliary equation and equating the coefficients of $\phi^j(\xi)\phi'(\xi)(0 \le j \le 5)$ to zero, a set of algebraic equations are yielded that unknown parameters $b_0(t), b_1(t), b_{-1}(t), k(t), v(t)$ are able to solve by using the computation of Maple.

We get two conditions:

Type 1:

$k(t) = H_5, c = c, a = -\frac{b_{-1}(t)(\alpha(t)+\beta(t))}{30k^2(t)}, b = b, \alpha(t) = -\beta(t) + k^2(t)H_2, \beta(t) = \beta(t), \gamma(t) = \frac{1}{10}\alpha(t)\beta(t) + \frac{1}{10}\alpha^2(t),$

$v(t) = \int(\alpha(t)b_0(t)k(t)^3 b + \frac{1}{10}b_0(t)^2 k(t)\alpha(t)\beta(t) + \frac{1}{10}b_0(t)^2 k(t)\alpha^2(t) + k^5(t)b^2)dt + H_1,$

$b_0(t) = H_4, b_1(t) = 0, b_{-1}(t) = H_3$

Type 2:
$k(t)=D_4, c=c, a=a, b=\frac{6b_0(t)a}{b_2(t)}, \alpha(t)=\alpha(t), \beta(t)=\beta(t), b_0(t)=D_3, b_1(t)=0, b_{-1}(t)=D_2,$

$\gamma(t)=\frac{-3\beta(t)b_{-1}(t)k^2(t)a-90k(t)^4a^2-6\alpha(t)b_{-1}(t)k(t)^2a}{b_{-1}(t)^2}$

We obtain analytical interaction solutions of eq(15) in type 1:

$$u_1(\xi)=H_4+\frac{H_3}{a_1A_1\frac{A_1C_1\tanh(\xi)+C_2\tan(\xi)}{A_1C_1+C_2\tan(\xi)\coth(\xi)}} \tag{22}$$

$$u_2(\xi)=H_4+\frac{H_3}{a_1A_1\frac{C_1\tan(\xi)+A_1C_2\coth(\xi)}{C_1\tanh(\xi)\tan(\xi)+A_1C_2}} \tag{23}$$

$$u_3(\xi)=H_4+\frac{H_3}{a_1A_1\frac{A_1C_1\tanh(\xi)+C_2\cot(\xi)}{A_1C_1+C_2\cot(\xi)\coth(\xi)}} \tag{24}$$

$$u_4(\xi)=H_4+\frac{H_3}{a_1A_1\frac{C_1\cot(\xi)+A_1C_2\coth(\xi)}{C_1\tanh(\xi)\cot(\xi)+A_1C_2}} \tag{25}$$

$$u_5(\xi)=H_4+\frac{H_3}{a_1A_1\frac{C_1+A_1C_2\tan(\xi)\coth(\xi)}{C_1\tanh(\xi)+A_1C_2\tan(\xi)}} \tag{26}$$

$$u_6(\xi)=H_4+\frac{H_3}{a_1A_1\frac{A_1C_1\tan(\xi)\tan(\xi)+C_2}{A_1C_1\tan(\xi)+C_2\coth(\xi)}} \tag{27}$$

$$u_7(\xi)=H_4+\frac{H_3}{a_1A_1\frac{A_1C_1\tanh(\xi)\cot(\xi)+C_2}{A_1C_1\cot(\xi)+C_2\coth(\xi)}} \tag{28}$$

$$u_8(\xi)=H_4+\frac{H_3}{a_1A_1\frac{C_1+A_1C_2\cot(\xi)\coth(\xi)}{C_1\tanh(\xi)+A_1C_2\cot(\xi)}} \tag{29}$$

$$u_9(\xi)=H_4+\frac{H_3}{a_0+\frac{a_1\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}{1-\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}+a_3(\exp(\frac{\xi}{2}))^2} \tag{30}$$

$$u_{10}(\xi)=H_4+\frac{H_3}{a_0+\frac{a_1\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}{1-\frac{\mathrm{sn}^2(\sqrt{2}\xi,\frac{\sqrt{2}}{2})}{2(1\pm\mathrm{dn}(\sqrt{2}\xi,\frac{\sqrt{2}}{2}))^2}}+a_3(\exp(-\frac{\xi}{2}))^2}, \tag{31}$$

Where $H_1, H_2, H_3, H_4, H_5, C_1, C_2, A_1$ are arbitrary constants. a_0, a_1, a_3 have the same meaning above.

We get solutions eq(22)-(29) which contain trigonometric functions and hyperbolic functions. Solutions of eq(30)-(31) include exponential functions, elliptic functions. They are all determined by solvable new arbitrary equation and they are novel interaction solutions of eq(19) which are not obtained in ref[1]. The phenomena of appearance of interaction waves are instantaneous and changeful. The interaction solutions are so complex that the influences of solutions are not easy to uncover. The effects are so significant, as nonlinear phenomena appear always everywhere in nature.

5. Conclusion and Discussion

In this paper, a new auxiliary equation is considered which we seek out multiple function solutions. A new auxiliary equation method is presented. Some new exact interaction solutions of the fifth-order KdV equation with variable coefficients are obtained by using this method. This method can be easily and effectively applied to other partial differential equations with constant or variable coefficients. It draws great attention that solutions of the novel auxiliary equation themselves include trigonometric function, hyperbolic function, elliptic function and other functions. It is not proposed in previous auxiliary equation methods. Complicate physical phenomena in nonlinear model systems will be described well when we analyze the typical interaction solutions we obtained in this paper.

ACKNOWLEDGEMENTS

This research is supported by Applied Mathematics Enhancement Program of Linyi University, the Natural Science Foundation of Shandong Province of China under Grant (ZR2010AM014) and the Scientific Research Fund from Shandong Provincial Education Department (J07WH04).

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