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Vertex-Edge Domination Polynomials of Product of Some Complete Graphs

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Abstract: *Let G = (V, E) be a simple Graph of order n. The vertex-edge domination polynomial of graph G is* $D_{ve}(G, x) = \sum_{i=\gamma_{ve}(G)}^{|V(G)|} d_{ve}(G, i)\, x^i$ *, where* $d_{ve}(G, i)$ *is the number of vertex-edge dominating sets of G of cardinality i and* $\gamma_{ve}(G)$ *is the vertex-edge domination number of G. In this paper we study the vertex-edge domination polynomials of product of some complete Graphs and some Interesting results are established.*

Keywords: *Vertex-edge dominating set, Vertex-edge domination polynomial, Vertex-edge dominating roots.*

1. Introduction

Let G = (V, E) be a simple graph of order n. A set $S \subseteq V$ is a dominating set of G, if every vertex in V\S is adjacent to atleast one vertex in S. The domination number of a graph, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G. A set of vertices in a Graph G is said to be a vertex-edge dominating set, if for all edges $e \in E(G)$, there exists a vertex $v \in S$ such that v dominates e. Otherwise, for a graph G = (V, E), a vertex $u \in V(G)$ ve - dominates an edge $vw \in E(G)$ if (i) u = v or u = w (u is incident to vw) or (ii) uv or uw is an edge in G (u is incident to an edge is adjacent to vw).

The minimum cardinality of a vertex-edge dominating set of G is called the vertex-edge domination number of G, and is denoted by $\gamma_{ve}(G)$.

Let G be a simple Graph of order n and let $d_{ve}(G, i)$ be the number of vertex-edge dominating sets of G with cardinality i. Then the vertex-edge domination polynomial $D_{ve}(G, x)$ of G is defined as :

$$D_{ve}(G, x) = \sum_{i=\gamma_{ve}(G)}^{|V(G)|} d_{ve}(G, i)\, x^i ,$$

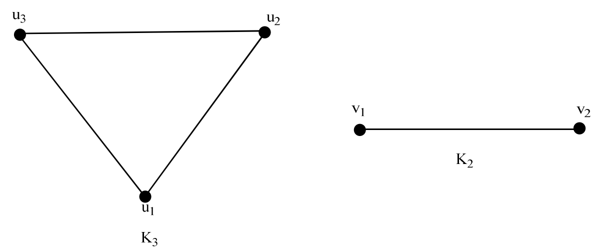
where $\gamma_{ve}(G)$ is the vertex-edge domination number of the Graph G. The roots of vertex-edge domination polynomial are called the vertex-edge domination roots of G. In the next section, we construct the families of the vertex-edge dominating sets of $K_r \,.\, K_2$. In section 3, we use the results obtained in section 2 to study the vertex-edge domination polynomial of $K_r \,.\, K_2$. In section 4, we study further results on vertex-edge domination polynomials of Graphs.

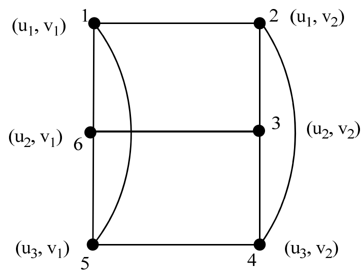
2. Vertex-Edge Dominating Sets of $K_R \,.\, K_2$

Definition: 2.1

Given any two Graphs G and H, we define the Cartesian product, denoted by G . H, to be a Graph with vertex set $V(G) \times V(H)$ and edges between two vertices (u_1, v_1) and (u_2, v_2) iff either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $u_1u_2 \in E(G)$ and $v_1 = v_2$.

Example: 2.2





$K_3 \cdot K_2$

$K_3 \cdot K_2 = \{ (u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2), (u_3, v_1), (u_3, v_2)\}$

No vertex-edge dominating set of $K_3 \cdot K_2$ is of cardinality 1

$\therefore d_{ve}(K_3 \cdot K_2 , 1) = 0$

Vertex-edge dominating sets of $K_3 \cdot K_2$ of cardinality 2 are

{{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {2, 3}, {2, 4}, {2, 5}, {2, 6}, {3, 4}, {3, 5}, {3, 6}, {4, 5}, {4, 6}, {5, 6}}

$\therefore d_{ve}(K_3 \cdot K_2 , 2) = 15$

Vertex-edge dominating sets of $K_3 \cdot K_2$ of cardinality 3 are

{{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6}, {1, 3, 4}, {1, 3, 5}, {1, 3, 6}, {1, 4, 5}, {1, 4, 6}, {1, 5, 6}, {2, 3, 4}, {2, 3, 5}, {2, 3, 6}, {2, 4, 5}, {2, 4, 6}, {2, 5, 6}, {3, 4, 5}, {3, 4, 6}, {3, 5, 6}, {4, 5, 6}}

$\therefore d_{ve}(K_3 \cdot K_2 , 3) = 20$

Vertex-edge dominating sets of $K_3 \cdot K_2$ of cardinality 4 are

{{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 3, 6}, {1, 2, 4, 5}, {1, 2, 4, 6}, {1, 2, 5, 6}, {1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6}, {1, 4, 5, 6}, {2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6}, {2, 4, 5, 6}, {3, 4, 5, 6}}

$\therefore d_{ve}(K_3 \cdot K_2 , 4) = 15$

Vertex-edge dominating sets of $K_3 \cdot K_2$ of cardinality 5 are

{{1, 2, 3, 4, 5}, {1, 2, 3, 4, 6}, {1, 2, 3, 5, 6}, {1, 2, 4, 5, 6}, {1, 3, 4, 5, 6}, {2, 3, 4, 5, 6}}

$\therefore d_{ve}(K_3 \cdot K_2 , 5) = 6$

Vertex-edge dominating sets of $K_3 \cdot K_2$ of cardinality 6 are

{{1, 2, 3, 4, 5, 6}}

$\therefore d_{ve}(K_3 \cdot K_2 , 6) = 1$

Theorem: 2.3

The vertex-edge dominating sets of $K_r \cdot K_2$ is

$$d_{ve}(K_r \cdot K_2 , n) = \begin{cases} \binom{2r}{n} - 2\binom{r}{n}, & n < r-1 \\ \binom{2r}{n} - 2\binom{r}{n} + 2r, & n = r-1 \\ \binom{2r}{n}, & n \geq r \end{cases}$$

Proof:

Since K_r has r vertices, $K_r \cdot K_2$ has 2r vertices, they are ordered pairs. Two of the vertices of $K_r \cdot K_2$ are enough to cover all the vertices and edges of $K_r \cdot K_2$. Therefore the minimum cardinality is 2. Therefore, $\gamma_{ve}(K_r \cdot K_2) = 2$.

$V(K_r) = \{u_1, u_2, \dots, u_r\}$ and $V(K_2) = \{v_1, v_2\}$. Let $n < r - 1$, $V(K_r \cdot K_2)$ consists of 2r vertices. Of these 2r vertices, n vertices ve-dominates $K_r \cdot K_2$, remaining $2\binom{r}{n}$ vertices are not ve-dominating sets.

Therefore, The number of vertex-edge dominating sets of $K_r \cdot K_2$ is $\binom{2r}{n} - 2\binom{r}{n}$.

if $n = r - 1$, then the number of vertex-edge dominating sets of $K_r \cdot K_2$ is $\binom{2r}{n} - 2\binom{r}{n} + 2r$.

Let $n \geq r$, any set of n vertices is a vertex-edge dominating set of $K_r \cdot K_2$. Therefore, the number of vertex-edge dominating sets of $K_r \cdot K_2$ is $\binom{2r}{n}$.

Hence

$$d_{ve}(K_r \cdot K_2, n) = \begin{cases} \binom{2r}{n} - 2\binom{r}{n}, & n < r-1 \\ \binom{2r}{n} - 2\binom{r}{n} + 2r, & n = r-1 \\ \binom{2r}{n}, & n \geq r \end{cases}.$$

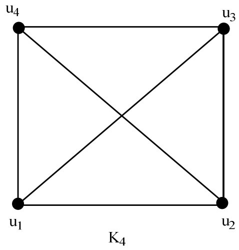
3. The Vertex-Edge Domination Polynomial of $K_R \cdot K_2$

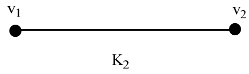
Definition: 3.1

Let $d_{ve}(K_r \cdot K_2, i)$ be the families of vertex-edge dominating sets of $K_r \cdot K_2$ with cardinality i. Then, the vertex-edge domination polynomial of $K_r \cdot K_2$ is

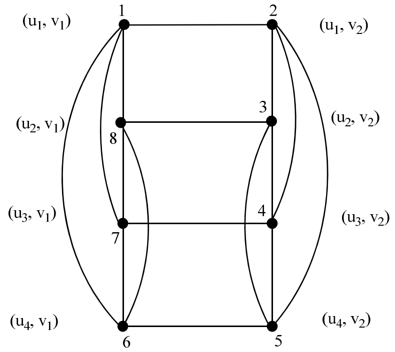
$$D_{ve}(K_r \cdot K_2, x) = = \sum_{i=\gamma_{ve}(K_r \cdot K_2)}^{|V(K_r \cdot K_2)|} d_{ve}(K_r \cdot K_2, i)\, x^i .$$

Example 3.2





$K_4 \cdot K_2 = \{ (u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2), (u_3, v_1), (u_3, v_2), (u_4, v_1), (u_4, v_2)\}$



$K_4 \cdot K_2$

No Vertex-Edge dominating set of $K_4 \cdot K_2$ with cardinality 1

$\therefore d_{ve}(K_4 \cdot K_2, 1) = 0$

Vertex-Edge dominating sets of $K_4 \cdot K_2$ with cardinality 2 are

{{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 6},{2, 7}. {2, 8}, {3, 6}, {3, 7}, {3, 8}, {4, 6}, {4, 7}, {4, 8}, {5, 6}, {5, 7}, {5, 8}}

$\therefore d_{ve}(k_4 \cdot k_2, 2) = 16$

Vertex-Edge dominating sets of $K_4 \cdot K_2$ with cardinality 3 are

{{1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 2, 6},{1, 2, 7}, {1, 2, 8}, {1, 3, 4}, {1, 3, 5}, {1, 3, 6}, {1, 3, 7}, {1, 3, 8}, {1, 4, 5}, {1, 4, 6}, {1, 4, 7}, {1, 4, 8}, {1, 5, 6}, {1, 5, 7}, {1, 5, 8}, {1, 6, 7}, {1, 6, 8}, {1, 7, 8}, {2, 3, 4}, {2, 3, 5}, {2, 3, 6}, {2, 3, 7}, {2, 3, 8}, {2, 4, 5}, {2, 4, 6}, {2, 4, 7}, {2, 4, 8}, {2, 5, 6}, {2, 5, 7}, {2, 5, 8}, {2, 6, 7}, {2, 6, 8}, {2, 7, 8}, {3, 4, 5}, {3, 4, 6}, {3, 4, 7}, {3, 4, 8}, {3, 5, 6}, {3, 5, 7}, {3, 5, 8}, {3, 6, 7}, {3, 6, 8}, {3, 7, 8}, {4, 5, 6}, {4, 5, 7}, {4, 5, 8}, {4, 6, 7}, {4, 6, 8}, {4, 7, 8}, {5, 6, 7}, {5, 6, 8}, {5, 7, 8}, {6, 7, 8}}

$\therefore d_{ve}(K_4 \cdot K_2, 3) = 56$

The number of Vertex-Edge dominating sets of $K_4 \cdot K_2$ with cardinality 4 is

$\therefore d_{ve}(K_4 \cdot K_2, 4) = 70$

The number of Vertex-Edge dominating sets of $K_4 \cdot K_2$ with cardinality 5 is

$$\therefore d_{ve}(K_4 \cdot K_2, 5) = \binom{8}{5} = 56$$

The number of Vertex-Edge dominating sets of $K_4 \cdot K_2$ with cardinality 6 are

$$\therefore d_{ve}(K_4 \cdot K_2, 6) = \binom{8}{6} = 28$$

The number of Vertex-Edge dominating sets of $K_4 \cdot K_2$ with cardinality 7 is

$$d_{ve}(K_4 \cdot K_2, 7) = \binom{8}{7} = 8$$

The number of Vertex-Edge dominating sets of $K_4 \cdot K_2$ with cardinality 8 are

$$d_{ve}(K_4 \cdot K_2, 8) = \binom{8}{8} = 1$$

$\therefore$ The Vertex-edge domination polynomial of $K_4 \cdot K_2$ is

$$\begin{aligned}
D_{ve}(K_4 \cdot K_2, x) &= \sum_{i=\gamma_{ve}(K_4 \cdot K_2)}^{|V(K_4 \cdot K_2)|} d_{ve}(K_4 \cdot K_2, i)\, x^i \\
&= \sum_{i=2}^{8} d_{ve}(K_4 \cdot K_2, i)\, x^i \\
&= d_{ve}(K_4 \cdot K_2, 2)\, x^2 + d_{ve}(K_4 \cdot K_2, 3)\, x^3 \\
&\quad + d_{ve}(K_4 \cdot K_2, 4)\, x^4 \\
&\quad + d_{ve}(K_4 \cdot K_2, 5)\, x^5 \\
&\quad + d_{ve}(K_4 \cdot K_2, 6)\, x^6 \\
&\quad + d_{ve}(K_4 \cdot K_2, 7)\, x^7 \\
&\quad + d_{ve}(K_4 \cdot K_2, 8)\, x^8 \\
&= 16\, x^2 + 56\, x^3 + 70\, x^4 + 56\, x^5 + 28\, x^6 + 8\, x^7 + 1\, x^8 \\
&= x^8 + 8x^7 + 28\, x^6 + 56\, x^5 + 70\, x^4 + 56\, x^3 + 16\, x^2
\end{aligned}$$

Vertex-edge domination polynomial of $K_3 \cdot K_2$ is

$$\begin{aligned}
D_{ve}(K_3 \cdot K_2, x) &= x^6 + 6x^5 + 15\, x^4 + 20\, x^3 + 15\, x^2 \\
&= ((1 + x)^3 - 1)^2 + 2x^3 + 6x^2
\end{aligned}$$

Vertex-edge domination polynomial of $K_4 \cdot K_2$ is

$$\begin{aligned}
D_{ve}(K_4 \cdot K_2, x) &= x^8 + 8\, x^7 + 28x^6 + 56x^5 + 70\, x^4 + 56\, x^3 + 16\, x^2 \\
&= ((1 + x)^4 - 1)^2 + 2x^4 + 8x^3
\end{aligned}$$

In general,

Vertex-edge domination polynomial of $K_r \,.\, K_2$ is

$$D_{ve}(K_r \,.\, K_2 , x) = ((1 + x)^r - 1)^2 + 2x^r + 2\,r\,x^{r-1}, \; r \geq 3.$$

Theorem 3.3

The vertex-Edge domination polynomial of $K_r \,.\, K_2$ is

$$D_{ve}(K_r \,.\, K_2, x) = ((1 + x)^r - 1)^2 + 2rx^{r-1} + 2x^r , \; r \geq 3.$$

Proof:

$$D_{ve}(K_r \,.\, K_2 , x) = \sum_{i=\gamma_{ve}(K_r \,.\, K_2)}^{|V(K_r \,.\, K_2)|} d_{ve}(K_r \,.\, K_2, i)\, x^i$$

$$= \sum_{i=2}^{2r} d_{ve}(K_r \,.\, K_2, i)\, x^i$$

$$= \sum_{i=2}^{r-2} d_{ve}(K_r \,.\, K_2, i)\, x^i + \sum_{i=r-1}^{r-1} d_{ve}(K_r \,.\, K_2, i)\, x^i + \sum_{i=r}^{2r} d_{ve}(K_r \,.\, K_2, i)\, x^i$$

$$= \sum_{i=2}^{r-2}\left[\binom{2r}{i} - 2\binom{r}{i}\right]x^i + \sum_{i=r-1}^{r-1}\left[\binom{2r}{i} - 2\binom{r}{i} + 2r\right]x^i + \sum_{i=r}^{2n}\binom{2r}{i}x^i$$

$$= \left[\binom{2r}{2} - 2\binom{r}{2}\right]x^2 + \left[\binom{2r}{3} - 2\binom{r}{3}\right]x^3 + \ldots$$

$$+ \left[\binom{2r}{r-2} - 2\binom{r}{r-2}\right]x^{r-2} + \left[\binom{2r}{r-1} - 2\binom{r}{r-1} + 2r\right]x^{r-1}$$

$$+ \binom{2r}{r}x^r + \binom{2r}{r+1}x^{r+1} + \ldots + \binom{2r}{2r}x^{2r}$$

$$= \binom{2r}{2}x^2 + \binom{2r}{3}x^3 + \ldots + \binom{2r}{r-2}x^{r-2} + \binom{2r}{r-1}x^{r-1}$$

$$+ \binom{2r}{r}x^r + \binom{2r}{r+1}x^{r+1} + \ldots + \binom{2r}{2r}x^{2r}$$

$$- 2\left[\binom{r}{2}x^2 + \binom{r}{3}x^3 + \ldots + \binom{r}{r-2}x^{r-2} + \binom{r}{r-1}x^{r-1}\right]$$

$$= (1 + x)^{2r} - \left[\binom{2r}{1}x + \binom{2r}{0}\right] - 2\left[\binom{r}{0} + \binom{r}{1}x + \ldots + \binom{r}{r-1}x^{r-1}\right.$$

$$\left. + \binom{r}{r}x^r\right] + 2x^r + 2\,r\,x^{r-1} + 2\left[\binom{r}{1}x + \binom{r}{0}\right]$$

$$= (1 + x)^{2r} - 2rx - 1 - 2(1 + x)^r + 2x^r + 2rx^{r-1} + 2rx + 2$$

$$= (1 + x)^{2r} - 2(1 + x)^r + 1 + 2x^r + 2\,r\,x^{r-1}$$

$$= [(1 + x)^r - 1]^2 + 2x^r + 2\,r\,x^{r-1}, \; r \geq 3.$$

4. Further Results on Vertex-Edge Domination Polynomial of Graphs

Theorem: 4.1

If G is a complete Graph of order n, then the vertex-edge dominating roots of $G \circ K_2$ are 0 with multiplicity n, $\omega - 1$ with multiplicity n, $\omega^2 - 1$ with multiplicity n.

Proof:

The vertex-edge dominating polynomial of $G \circ K_2$ is

$$D_{ve}(G \circ K_2 , x) = [(1 + x)^3 - 1]^n$$

To find the roots, put $D_{ve}(G \circ K_2, x) = 0$

$$[(1+x)^3 - 1]^n = 0$$

$$\Rightarrow [(1+x)^3 - 1]\,[(1+x)^3 - 1] \ldots [(1+x)^3 - 1] \text{ n times} = 0$$

$$\Rightarrow \quad (1+x)^3 - 1 = 0,\ (1+x)^3 - 1 = 0 \ \ldots (1+x)^3 - 1 = 0$$

$$\Rightarrow \quad (1+x)^3 = 1$$

$$\therefore \quad 1 + x = 1^{1/3}$$

$$= 1, \omega, \omega^2 \text{ where } \omega = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\therefore \quad x = 1 - 1, \omega - 1, \omega^2 - 1$$

$$= 0, \omega - 1, \omega^2 - 1$$

$\therefore$ The vertex-edge dominating roots of $G \circ K_2$ are 0 with multiplicity n, $\omega - 1$ with multiplicity n, $\omega^2 - 1$ with multiplicity n.

Proposition 4.2

If G is a complete Graph of order n, then $D_{ve}(G \circ K_2, -1) = (-1)^n$

Proof:

We know that the vertex-edge dominating polynomial of $G \circ K_2$ is

$$D_{ve}(G \circ K_2, x) = [(1+x)^3 - 1]^n$$

$$\therefore D_{ve}(G \circ K_2, -1) = [(1-1)^3 - 1]^n$$

$$= (0 - 1)^n$$

$$= (-1)^n$$

Theorem: 4.3

The vertex-edge dominating roots of the star graph (S_n) are

$$\frac{\cos 2(k+1)\pi}{n} + \frac{i \sin 2(k+1)\pi}{n} - 1,\ i = 0, 1, 2, \ldots, n-1.$$

Proof:

The vertex-edge dominating polynomial of star graph (S_n) is

$$D_{ve}(S_n, x) = (1+x)^n - 1$$

To find the vertex-edge dominating roots, put $D_{ve}(S_n, x) = 0$

$$(1+x)^n - 1 = 0$$

$$\therefore \quad (1+x)^n = 1$$

$$(1+x) = 1^{1/n}$$

$$= (\cos 2\pi + i \sin 2\pi)^{1/n}$$

$$= [\cos(2k\pi + 2\pi) + i\sin(2k\pi + 2\pi)]^{1/n} \text{ where k is an integer}$$

$$= [\cos 2(k+1)\pi + i \sin 2(k+1)\pi]^{1/n},\ k = 0, 1, 2, \ldots, n-1$$

$$1 + x = \frac{\cos 2(k+1)\pi}{n} + \frac{\sin 2(k+1)\pi}{n},\ k = 0, 1, 2, \ldots, n-1$$

$$x = \frac{\cos 2(k+1)\pi}{n} + \frac{i \sin 2(k+1)\pi}{n} - 1,\ k = 0, 1, 2, \ldots, n-1.$$

$\therefore$ The vertex-edge dominating roots of the star Graph S_n are

$$\frac{\cos 2(k+1)\pi}{n} + \frac{i \sin 2(k+1)\pi}{n} - 1,\ k = 0, 1, 2, \ldots, n-1.$$

Result : 4.4

$$\frac{d^n}{dx^n} D_{ve}(S_n, x) = n!$$

Proof:

The vertex-edge dominating polynomial of star graph (S_n) is

$$D_{ve}(S_n, x) = (1+x)^n - 1$$

D.w.r.to x,

$$\frac{d}{dx} D_{ve}(S_n, x) = n (1 + x)^{n-1}$$

D.w.r.to r,

$$\frac{d^2}{dx^2} D_{ve}(S_n, x) = n (n - 1) (1 + x)^{n-2}$$

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$$\frac{d^n}{dx^n} D_{ve}(S_n, x) = n(n - 1) (n - 2) \ldots (n - (n - 1)) (1 + x)^{n-n}$$

$$= n (n - 1) (n - 2) \ldots 1$$

$$= n!.$$

Proposition: 4.5

Let S_n is the star graph with n vertices $D_{ve}(S_n, -1) = -1$.

Proof:

The vertex-edge domination polynomial of star graph (S_n) is

$$D_{ve}(S_n, x) = (1 + x)^n - 1$$

$$\therefore D_{ve}(S_n, -1) = (1 - 1)^n - 1$$

$$= 0 - 1$$

$$= - 1$$

5. CONCLUSION

The vertex-edge domination polynomial of a graph is one of the algebraic representations of the Graph. This paper introduces vertex-edge domination polynomial of product of some complete Graphs. Similarly we can find the vertex-edge dominating sets and vertex-edge domination polynomials of some specified Graphs.

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