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Properties of the Lowest Common Ancestor in a Complete Binary Tree

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Abstract: *The paper presents and proves several theorems that disclose kinds of relations between a lowest-common-ancestor (LCA) and its descendant nodes in a complete binary tree. The proved theorems involve in determination of the LCA of two nodes on the same level, the LCA of two nodes on the same path, the LCA of three nodes on the same level and the LCA of arbitrary two nodes in a complete binary tree. Through strict mathematical deductions, conditions in analytic formulas and equations are given for computation of the LCAs.*

Keywords: *Discrete mathematics, Data structure, Complete binary tree, Lowest common ancestor, Algorithm design*

1. Introduction

Problem of finding the lowest common ancestor (LCA) of two nodes in a tree was first raised in 1973 by Alfred Aho, John Hopcroft, and Jeffrey Ullman. As a common fundamental problem in both graph theorem and computer science, the problem has been widely focused as described in bibliographies [1]-[3]. People have designed several algorithms to solve the problem, as introduced in bibliographies [4]-[6]. Seeking the LCA of two or more nodes in a complete binary tree, as a special case of the previous problem, has recently paid attentions in bioinfomatics and industrial applications in pattern matching [7]-[9], rapid location of industrial data flow [10]-[12] and development of system on chip (SoC) [13]-[15].

Look trough present bibliographies of computing an LCA, e.g., the bibliographies [4]-[6], one can see that most bibliographies only pay their attention to finding an LCA of two or more nodes, and few concern the LCAs' properties related to their descendant nodes. Actually, there has not been a bibliography that mathematically presents a complete descripts about an LCA's property. This leaves a bug for us to analyze LCAs, especially the LCAs from three or more nodes.

In designing new and fast algorithms for traversal of a binary tree, as illustrated in papers [13], [14] and [15], I find it necessary to know clearly the relations between an LCA and its descendant nodes. Thus I have made a systematic study on the issue. This paper presents my research results. Through strict mathematical deductions, I have proved several theorems that disclose kinds of relations between an LCA and its descendant nodes in a complete binary tree. Section 3 will show the theorems and their proofs.

2. Preliminaries

We need the following lemmas and notations for later sections.

2.1 Definitions and Symbols

Definitions related with binary trees can be seen by certain entries in book [16]. In this paper, we assume the depth of the complete binary tree we study is *h* and all the nodes we investigate are valid.

We use symbol $N_{(k,j)}$ to express the node at the *j*-th position on the *k*-th level (*k*>0) of a binary tree, $G_{(k,i)}^{(l,j)}$ to express the LCA of $N_{(k,i)}$ and $N_{(l,j)}$, and $l(N)$ to be number of the level on which the node N lies, for example, $l(N_{(k,j)})=k$. Symbol $\lfloor x \rfloor$ is to express the floor function defined by $x-1<\lfloor x \rfloor \le x$, where x is a real number.

2.2 Lemmas

Lemma1 ([16]). If each node of a complete binary tree that has n nodes is encoded by natural number 1,2, n by the way from top to bottom and from left to right, then the father node of node j (j>1) is $\lfloor j/2 \rfloor$; The k-th (k≥1) level of a complete binary tree has at most 2^{k-1} nodes; the code of the first node on the k-th level is 2^{k-1}.

Lemma2 ([17]). For any real x and integers n: $n\lfloor x \rfloor \le \lfloor nx \rfloor$; for any integer n and real x: $\lfloor n+x \rfloor = n+\lfloor x \rfloor$.

Lemma3 ([18]). Let α and δ be real number such that $\delta>0$ and $\alpha \ge 0$ and χ be integers such that $\chi=,\ldots,-3,-2,-1,0,1,2,3,\ldots,$; suppose that I is the smallest positive integer that fit the inequality $0\le \alpha \bmod 2^x < 2^{x-1}$ of unknown x, then when $(\chi-\frac{1}{2})\times 2^I < \delta \le (\chi+\frac{1}{2})\times 2^I$ it holds $\left\lfloor \frac{\alpha+\delta}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi$

Lemma4 ([19]). For any real x and y, $x \le y$ yields $\lfloor x \rfloor \le \lfloor y \rfloor$ and $x \ge y$ yields $\lfloor x \rfloor \ge \lfloor y \rfloor$.

Lemma5. Suppose that a, b are positive real numbers such that $a \ge b$; if integers x, y fit the equality $\left\lfloor \frac{a}{2^x} \right\rfloor = \left\lfloor \frac{b}{2^y} \right\rfloor$, then $x \ge y$.

Proof. We adapt a proof by contradiction. Assume $x<y$; then $x \le y+1$, which leads to $\frac{a}{2^x} \ge \frac{2a}{2^y} \ge \frac{2b}{2^y}$. By Lemma 2 and Lemma 4, it yields $\left\lfloor \frac{a}{2^x} \right\rfloor \ge \left\lfloor 2\frac{b}{2^y} \right\rfloor \ge 2\left\lfloor \frac{b}{2^y} \right\rfloor = 2\left\lfloor \frac{a}{2^y} \right\rfloor$, which leads to a contradiction that $1 \ge 2$. Hence $x \ge y$.

3. Main Results and Proofs

We obtain theorems and propositions that includes the following aspects:

(1) LCA of two nodes on the same level;

(2) LCA of two nodes on the same path;

(3) LCA of three nodes on the same level;

(4) LCA of arbitrary two nodes in a complete binary tree.

Proposition1. The direct ancestors of $N_{(k,\alpha)}$ are $N_{(k-i,\sigma(i))}$, where $\sigma(i)=\left\lfloor \frac{\alpha}{2^i} \right\rfloor (i>0)$.

Proof. By Lemma 1, $N_{(k-1,\lfloor \alpha/2 \rfloor)}$ is the fathers of $N_{(k,\alpha)}$, and so forth, the direct ancestors of $N_{(k,\alpha)}$ are $N_{(k-i,\lfloor \alpha/2^i \rfloor)}$.

Theorem1. Let $N_{(k,\alpha)}$ and $N_{(k,\beta)}$ $(1<k\le h, 2^{k-1}\le \alpha < \beta < 2^k)$ be two nodes on the k-th level of a complete binary tree; if I is the smallest positive integer that fits the equation $\left\lfloor \frac{\alpha}{2^i} \right\rfloor = \left\lfloor \frac{\beta}{2^i} \right\rfloor$ of integer unknown i and $\sigma = \left\lfloor \frac{\alpha}{2^I} \right\rfloor$, then $N_{(k-I,\sigma)}$ is the LCA of $N_{(k,\alpha)}$ and $N_{(k,\beta)}$.

Proof. By Proposition 1, the direct ancestors of $N_{(k,\alpha)}$ and $N_{(k,\beta)}$ are respectively $N_{(k-i,\lfloor \alpha/2^i \rfloor)}$ and $N_{(k-i,\lfloor \beta/2^i \rfloor)}$ when $i>0$. Therefore $N_{(k-i,\varpi)}$ which satisfies $\left\lfloor \frac{\alpha}{2^i} \right\rfloor = \left\lfloor \frac{\beta}{2^i} \right\rfloor$ is a direct ancestor of

$N_{(k,\alpha)}$ and $N_{(k,\beta)}$. Hence when $I = \min(i)$ and $\sigma = \left\lfloor \frac{\alpha}{2^I} \right\rfloor = \left\lfloor \frac{\beta}{2^I} \right\rfloor$, $N_{(k-I,\sigma)}$ is the LCA of $N_{(k,\alpha)}$ and $N_{(k,\beta)}$.

Theorem 2. Let $N_{(k,\alpha)}$ and $N_{(k,\beta)}$ $(1 < k \le h, 2^{k-1} \le \alpha < \beta < 2^k)$ be two nodes on the *k*-th level of a complete binary tree; if *I* is the smallest positive integer solution that fits the inequality $0 \le \alpha \bmod 2^i < 2^{i-1}$ of unknown *i*, and σ, χ respectively satisfy $\sigma = \left\lfloor \frac{\alpha}{2^I} \right\rfloor$,

$$(\chi - \frac{1}{2}) \times 2^I < \beta - \alpha \le (\chi + \frac{1}{2}) \times 2^I$$

Then $N_{(k,\alpha)}$ and $N_{(k,\beta)}$ share their LCA with $N_{(k-I,\sigma)}$ and $N_{(k-I,\sigma+\chi)}$, namely, $G_{(k,\alpha)}^{(k,\beta)} = G_{(k-I,\sigma)}^{(k-I,\sigma+\chi)}$

Proof. Substituting δ by β-α in Lemma 3 immediately yields

$$\left\lfloor \frac{\beta}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi = \sigma + \chi$$

By Proposition 1, $N_{(k-I,\sigma)}$ is a direct ancestor of $N_{(k,\alpha)}$ and $N_{(k-I,\sigma+\chi)}$ is a direct ancestor of $N_{(k,\beta)}$; hence, the LCA of $N_{(k-I,\sigma)}$ and $N_{(k-I,\sigma+\chi)}$ must be the LCA of $N_{(k,\alpha)}$ and $N_{(k,\beta)}$.

Theorem 2 immediately induces the following proposition 2, which is shown in paper [20], propositions 3 and 4.

Proposition2. If *I* is the smallest positive integer solution of the inequality $0 \le \alpha \bmod 2^i < 2^{i-1}$ of unknown *i* and $\sigma = \left\lfloor \frac{j}{2^I} \right\rfloor$, then $N_{(k-I,\sigma)}$ is the LCA of $N_{(k,j)}$ and $N_{(k,j+1)}$.

Proof. Taking $\alpha = j$ and $\beta = \alpha + 1$ in Theorem 2 immediate yields $\chi = 0$.

Proposition3. Let $N_{(k,\alpha)}$ and $N_{(k,\gamma)}$ $(1 < k \le h, 2^{k-1} \le \alpha < \gamma < 2^k)$ be two nodes on the *k*-th level of a complete binary tree; then for arbitrary β such that $\alpha \le \beta \le \gamma$, it holds $l(G_{(k,\alpha)}^{(k,\gamma)}) \le l(G_{(k,\alpha)}^{(k,\beta)})$.

Proof. Let $\delta_1 = \beta - \alpha \ge 1$ and $\delta_2 = \gamma - \alpha \ge 1$; suppose *I* is the smallest positive integer solution of the inequality $0 \le \alpha \bmod 2^i < 2^{i-1}$ of unknown *i*; then by Lemma 3 it holds

$$\left\lfloor \frac{\alpha + \delta_1}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi_1$$

$$\left\lfloor \frac{\alpha + \delta_2}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi_2$$

Where χ_1 and χ_2 are integers.

Since $\alpha \le \beta \le \gamma$, we know $\delta_2 \ge \delta_1$ and therefore $\chi_2 \ge \chi_1$.

Let $\sigma = \left\lfloor \frac{j}{2^I} \right\rfloor$, then by Proposition 1 we know that the direct ancestors of $N_{(k,\alpha)}$, $N_{(k,\beta)}$ and $N_{(k,\gamma)}$ are respectively $N_{(k-I,\alpha)}$, $N_{(k-I,\alpha+\chi_1)}$ and $N_{(k-I,\alpha+\chi_2)}$. By Theorem 2 it yields

$$G_{(k,\alpha)}^{(k,\beta)} = G_{(k-I,\sigma)}^{(k-I,\sigma+\chi_1)}, \quad G_{(k,\alpha)}^{(k,\gamma)} = G_{(k-I,\sigma)}^{(k-I,\sigma+\chi_2)}$$

Now let *X* and *Y* be respectively the smallest positive integer solutions of the following equations of unknown *i* and *j*

$$\left\lfloor \frac{\sigma}{2^i} \right\rfloor = \left\lfloor \frac{\sigma + \chi_1}{2^i} \right\rfloor, \left\lfloor \frac{\sigma}{2^j} \right\rfloor = \left\lfloor \frac{\sigma + \chi_2}{2^j} \right\rfloor$$

then by Theorem 1, we know

$G_{(k,\alpha)}^{(k,\beta)} = G_{(k-I,\sigma)}^{(k-I,\sigma+\chi_1)} = N_{(k-I-X,\lfloor \sigma/2^X \rfloor)} = N_{(k-I-X,\lfloor \sigma/2^{I+X} \rfloor)}$, $G_{(k,\alpha)}^{(k,\gamma)} = G_{(k-I,\sigma)}^{(k-I,\sigma+\chi_2)} = N_{(k-I-Y,\lfloor \sigma/2^Y \rfloor)} = N_{(k-I-Y,\lfloor \sigma/2^{I+Y} \rfloor)}$

and

$$\left\lfloor \frac{\sigma+\chi_1}{2^X} \right\rfloor = \left\lfloor \frac{\sigma+\chi_2}{2^Y} \right\rfloor$$

By Lemma 5,we know $Y \ge X$, namely,

$l(G_{(k,\alpha)}^{(k,\gamma)}) = l(N_{(k-I-X,\lfloor \sigma/2^X \rfloor)}) \le l(G_{(k,\alpha)}^{(k,\beta)}) = l(N_{(k-I-Y,\lfloor \sigma/2^{I+Y} \rfloor)})$.

Proposition4. Let $N_{(m,\alpha)}$ and $N_{(m,\gamma)}$ $(1 < k \le h, 2^{m-1} \le \alpha < \gamma < 2^m)$ be two nodes in a complete binary tree, then for arbitrary β such that $\alpha \le \beta \le \gamma$ it holds $l(G_{(m,\alpha)}^{(m,\gamma)}) \le l(G_{(m,\beta)}^{(m,\gamma)}$.

Proof. Let I, J, K be the smallest positive integers that respectively fit the following three equations of unknowns i, j, k

$$\left\lfloor \frac{\alpha}{2^i} \right\rfloor = \left\lfloor \frac{\beta}{2^i} \right\rfloor \tag{1}$$

$$\left\lfloor \frac{\beta}{2^j} \right\rfloor = \left\lfloor \frac{\gamma}{2^j} \right\rfloor \tag{2}$$

$$\left\lfloor \frac{\alpha}{2^k} \right\rfloor = \left\lfloor \frac{\gamma}{2^k} \right\rfloor \tag{3}$$

As Theorem 1 says, $G_{(m,\alpha)}^{(m,\beta)}$, $G_{(m,\beta)}^{(m,\gamma)}$ and $G_{(m,\alpha)}^{(m,\gamma)}$ are respectively determined by I, J, K . We use a proof by contradiction to show $J \le K$. First by Proposition 3 it yields $K \ge I$. Now we assume $J > K$; then it derives $J > I$. By this and (1), (2), it leads to

$$\left\lfloor \frac{\alpha}{2^J} \right\rfloor = \left\lfloor \frac{\beta}{2^J} \right\rfloor = \left\lfloor \frac{\gamma}{2^J} \right\rfloor$$

which is contrary to the fact that K is the smallest positive integer that fit (3). Hence $J \le K$. Since $l(G_{(m,\alpha)}^{(m,\gamma)}) = m - K$ and $l(G_{(m,\beta)}^{(m,\gamma)}) = m - J$, it knows $l(G_{(m,\alpha)}^{(m,\gamma)}) \le l(G_{(m,\beta)}^{(m,\gamma)}$.

Theorem3. Let $N_{(k,i)}$ and $N_{(l,j)}$ $(1 < k < l \le h, 2^{k-1} \le i < 2^k, 2^{l-1} \le j < 2^l)$ be two nodes in a complete binary tree; then the two share a common direct ancestor (or on the same path) if and only if there exists a positive integer σ such that

$$i = \left\lfloor \frac{j}{2^\sigma} \right\rfloor, k = l - \sigma \tag{4}$$

and then $N_{(k-1,\lfloor i/2 \rfloor)}$ is the LCA of $N_{(k,i)}$ and $N_{(l,j)}$.

Proof. We first prove the first conclusion by its sufficiency and necessity.

Sufficiency. By the proposition 1, $N_{(l-\sigma,\lfloor j/2^\sigma \rfloor)}$ is the σ^{th}-generation direct ancestor of the $N_{(l,j)}$. This fact and the condition (4) show that $N_{(k,i)}$ is a direct ancestor of $N_{(l,j)}$, and thus the two nodes share a common direct ancestor. Hence the sufficiency holds.

Necessity. If $N_{(k,i)}$ and $N_{(l,j)}$ has the same direct ancestor, then $k < l$ leads to that $N_{(k,i)}$ is an ancestor of $N_{(l,j)}$. By property of complete binary trees, a sub-tree with root $N_{(k,i)}$ has at most 2^σ nodes on its σ-th level (or on its maternal tree's $k + \sigma = l$ -th level). The possible positions of these nodes in the maternal tree are

$$2^\sigma i, 2^\sigma i + 1, 2^\sigma i + 2, ..., 2^\sigma i + 2^\sigma - 1 \tag{5}$$

Since $N_{(l,j)}$ is a descendant of $N_{(k,i)}$ and $k + \sigma = l$, we know that j must be one of (5). Considering that an arbitrary λ such that $0 \le \lambda < 2^{\sigma-1}$ will lead to

$\left\lfloor \frac{2^\sigma i+\lambda}{2^\sigma} \right\rfloor = \left\lfloor i+\frac{\lambda}{2^\sigma} \right\rfloor = i$ (By Lemma 2)

This implies $\left\lfloor \frac{j}{2^\sigma} \right\rfloor = i$, which validates the necessity. Hence the first conclusion holds.

Since $N_{(k,i)}$ is an ancestor of $N_{(l,j)}$, its father, $N_{(k-1,\lfloor i/2 \rfloor)}$, is certainly the LCA of $N_{(k,i)}$ and $N_{(l,j)}$.

Theorem 4. Let $N_{(m,\alpha)}$ and $N_{(m,\gamma)}$ $(1<m\le h; 2^{m-1}\le\alpha<\gamma<2^m)$ be two nodes in a complete binary tree; then for arbitrary β such that $\alpha\le\beta\le\gamma$, it holds

$$l(G_{(m,\alpha)}^{(m,\gamma)}) = \min(l(G_{(m,\alpha)}^{(m,\beta)}), l(G_{(m,\beta)}^{(m,\gamma)})) \tag{6}$$

Proof. By definition, $G_{(m,\alpha)}^{(m,\beta)})$ is a direct ancestor of $N_{(m,\alpha)}$ and $N_{(m,\beta)}$, and $G_{(m,\beta)}^{(m,\gamma)}$ is a direct ancestor of $N_{(m,\beta)}$ and $N_{(m,\gamma)}$; namely, both $G_{(m,\alpha)}^{(m,\beta)})$ and $G_{(m,\beta)}^{(m,\gamma)}$ are direct ancestor of $N_{(m,\beta)}$. Hence the one who lies in upper level is a direct ancestor of the other, and obviously is a common ancestor of $N_{(m,\alpha)}$, $N_{(m,\beta)}$ and $N_{(m,\gamma)}$. Consequently, if $G_{(m,\alpha)}^{(m,\gamma)}$ is the LCA of $N_{(m,\alpha)}$ and $N_{(m,\gamma)}$, then

$$l(G_{(m,\alpha)}^{(m,\gamma)}) \ge \min(l(G_{(m,\alpha)}^{(m,\beta)}), l(G_{(m,\beta)}^{(m,\gamma)})) \tag{7}$$

By the proposition 3 and 4, it simultaneously holds

$$l(G_{(m,\alpha)}^{(m,\gamma)}) \le l(G_{(m,\alpha)}^{(m,\beta)}),\ l(G_{(m,\alpha)}^{(m,\gamma)}) \le l(G_{(m,\beta)}^{(m,\gamma)})$$

So that it yields

$$l(G_{(m,\alpha)}^{(m,\gamma)}) = \min(l(G_{(m,\alpha)}^{(m,\beta)}), l(G_{(m,\beta)}^{(m,\gamma)})).$$

Proposition 5.Let $N_{(m,\alpha)}$ and $N_{(n,\beta)}$ $(1<m\le n\le h, 2^{m-1}\le\alpha<2^m, 2^{n-1}\le\beta<2^n)$ be two nodes in a complete binary tree and $\sigma=n-m, \gamma=\left\lfloor \frac{\beta}{2^\sigma} \right\rfloor$, then $G_{(m,\alpha)}^{(m,\gamma)} = G_{(m,\alpha)}^{(n,\beta)}$.

Proof. By Theorem 3, $N_{(n-\sigma,\gamma)} = N_{(m,\gamma)}$ is an *m*-level-laid direct ancestor of $N_{(n,\beta)}$; therefore all the direct ancestors of $N_{(m,\gamma)}$ must be direct ancestors of $N_{(n,\beta)}$. Hence $G_{(m,\alpha)}^{(m,\gamma)} = G_{(m,\alpha)}^{(n,\beta)}$

4. CONCLUSION

The proved 4 theorems together with their 5 propositions form a complete system to describe relations among LCAs and their descendant nodes in a complete binary tree. Theorem 1 establishes a general condition to determine the LCA of two nodes, theorem 2 shows the relation between the LCA and direct ancestors of two nodes, theorem 3 depicts computation of the LCA of two nodes on the same path, propositions 3, 4 and theorem 4 give the relations of two LCAs among three nodes, and proposition 5 shows how to determine an LCA of arbitrary two nodes. With these theorems and propositions, one can easily know and determine the LCA of two nodes in a complete binary tree. Since a complete binary tree is an important content of discrete mathematics and graph theory, the study of this article can be a reference to discrete mathematics and graph theory.

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