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Another Characterization's of 2-Pre-Hilbert Space

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Abstract: *The problem of determination necessary and sufficient conditions for a* 2*-normed space to be* 2*-pre-Hilbert space is in the focus of interest of many mathematicians. Some characterizations of* 2*-inner product are stated in* [1], [4], [6], [7] *and* [12]*. In this paper we gave a necessary and sufficient condition for the existence of* 2*-inner product in a* 2*-normed space* $(L,\|\cdot,\cdot\|)$ *applying Mercer inequality and also the generalizations of Tanaka, Kirk-Smiley and Gurarii-Sozonov results are given*.

Keywords: *2-norm, 2-inner product, Dunkl-Williams inequality*

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1. INTRODUCTION

Concepts of 2-norm and 2-inner product are two-dimensional analogies of the concepts of norm and inner product, respectively. The studding of 2-normed spaces and their application is simpler if 2-norm is generated by 2-inner product. On the other hand, verifying whether 2-norm is generated by a 2-inner product is not always simple. Therefore, of particular importance is the finding of different equivalent conditions of existence of 2-inner product that generates 2-norm, which is of interest in this paper.

Let L be a real vector space with dimension greater than 1 and $\|\cdot,\cdot\|$ be a real function of $L\times L$ such that following holds true:

a) $\|x,y\|\geq 0$, for each $x,y\in L$ and $\|x,y\|=0$ if and only if the set $\{x,y\}$ is linearly dependent;
b) $\|x,y\|=\|y,x\|$, for each $x,y\in L$;
c) $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$, for each $x,y\in L$ and for each $\alpha\in\mathbf{R}$;
d) $\|x+y,z\|\leq\|x,z\|+\|y,z\|$, for each $x,y,z\in L$.

The function $\|\cdot,\cdot\|$ is said to be 2-*norm on L*, and $(L,\|\cdot,\cdot\|)$ is said to be *vector* 2-*norm space* ([8]). The above inequality *d*) is said to be *parallelepiped inequality*.

Let $n>1$ be a positive integer, L be a real vector space, $\dim L\geq n$ and $(\cdot,\cdot|\cdot)$ be a real function of $L\times L\times L$ such that

i) $(x,x|y)\geq 0$, for each $x,y\in L$ and $(x,x|y)=0$ if and only if x and y are linearly dependent;
ii) $(x,y|z)=(y,x|z)$, for each $x,y,z\in L$;
iii) $(x,x|y)=(y,y|x)$, for each $x,y\in L$;
iv) $(\alpha x,y|z)=\alpha(x,y|z)$, for each $x,y,z\in L$. and for each $\alpha\in\mathbf{R}$; and
v) $(x+x_1,y|z)=(x,y|z)+(x_1,y|z)$, for each $x_1,x,y,z\in L$.

Function $(\cdot,\cdot\,|\,\cdot)$ is said to be 2-*inner product*, and $(L,(\cdot,\cdot\,|\,\cdot))$ is said to be 2-*pre-Hilbert space* ([4]).

R. Ehret proved that ([7]), if $(L,(\cdot,\cdot\,|\,\cdot))$ is a 2-pre-Hilbert space, then

$$\| x, y \| = (x, x\,|\,y)^{1/2}, \tag{1}$$

for each $x, y \in L$ defines a 2-norm, so, we get vector 2-normed space $(L, \|\cdot,\cdot\|)$ and thus for each $x, y, z \in L$ the following equalities hold true

$$(x, y\,|\,z) = \frac{\|x+y,z\|^2 - \|x-y,z\|^2}{4}, \tag{2}$$

$$\| x+y, z \|^2 + \| x-y, z \|^2 = 2(\| x, z \|^2 + \| y, z \|^2). \tag{3}$$

The equality (3) in fact is an analogy of parallelogram equality and is said to be *parallelepiped equality*. Further, 2-normed space L is 2-pre-Hilbert if and only if for each $x, y, z \in L$ the equality (3) holds true. The following three Lemmas we will present a three elementary statements according to the equalities (3) and (4).

Lemma 1. Let $(L, \|\cdot,\cdot\|)$, $\dim L > 2$ be 2-normed space. If there exists a 2-inner product $(\cdot,\cdot\,|\,\cdot)_Y$, for each three-dimensional subspace Y of L, such that $(y, y\,|\,z)_Y = \| y, z \|^2$, for each $y, z \in Y$, then exists a 2-inner product $(\cdot,\cdot\,|\,\cdot)_L$ such that $(x, x\,|\,z)_L = \| x, z \|^2$, for each $x, z \in L$.

Proof. Let $x, y, z \in L$. Then it exists a subspace Y of L such that $\dim Y = 3$ and $x, y, z \in Y$. The assumption implies that it exists 2-inner product $(\cdot,\cdot\,|\,\cdot)_Y$ such that $(a, a\,|\,b)_Y = \| a, b \|^2$, for each $a, b \in Y$. But the last in fact means that the following holds true

$$\begin{aligned}\| x+y, z \|^2 + \| x-y, z \|^2 &= (x+y, x+y\,|\,z)_Y + (x-y, x-y)_Y\\ &= 2(x, x\,|\,z)_Y + 2(y, y\,|\,z)_Y\\ &= 2(\| x, z \|^2 + \| y, z \|^2).\end{aligned}$$

Finally, the arbitrariness of $x, y, z \in L$ implies that the parallelepiped equality holds in L. It means that L is 2-pre-Hilbert space, i.e. there exists 2-inner product $(\cdot,\cdot\,|\,\cdot)_L$ such that $(x, x\,|\,z)_L = \| x, z \|^2$, for each $x, z \in L$.

Lemma 2. Let $(\cdot,\cdot\,|\,\cdot)$ be 2-inner product in vector space L and let a linear mapping $T : L \to L$ be injection. Then,

$$(x, y\,|\,z)_T = (T(x), T(y)\,|\,T(z)), \quad x, y, z \in L \tag{4}$$

defines 2-inner product in L.

Proof. Let a linear mapping $T : L \to L$ be injection, and $(\cdot,\cdot\,|\,\cdot)_T$ be defined by (4). Then the following holds true

$$(x, x\,|\,z)_T = (T(x), T(x)\,|\,T(z)) \geq 0, \text{ for each } x, z \in L$$

and furthermore $(x, x\,|\,z)_T = 0$ if and only if $T(x)$ and $T(z)$ are linearly dependent, i.e. there exists $\alpha, \beta \in \mathbf{R}$ such that $\alpha \neq 0$ or $\beta \neq 0$ and $\alpha T(x) + \beta T(z) = 0$. The last actually means that $T(\alpha x + \beta z) = 0$, and since T is injection, and $T(0) = 0$ we conclude that $\alpha x + \beta z = 0$. According to that, $(x, x\,|\,z)_T = 0$ if and only if x and z are linearly dependent, i.e. the axiom *i*) of 2-inner product definition holds true.

Hence, T is linear mapping, and therefore the properties of 2-inner product imply that for each all $x, x_1, y, z \in L$ and for each $\alpha \in \mathbf{R}$ the following holds true

$$(x, y\,|\,z)_T = (T(x), T(y)\,|\,T(z)) = (T(y), T(x)\,|\,T(z)) = (y, x\,|\,z)_T,$$

$$(x,x\,|\,y)_T=(T(x),T(x)\,|\,T(y))=(T(y),T(y)\,|\,T(x))=(y,y\,|\,x)_T\,,$$

$$\begin{aligned}(\alpha x,y\,|\,z)_T&=(T(\alpha x),T(y)\,|\,T(z))=(\alpha T(x),T(y)\,|\,T(z))\\&=\alpha(T(x),T(y)\,|\,T(z))=\alpha(x,y\,|\,z)_T\,,\end{aligned}$$

$$\begin{aligned}(x+x_1,y\,|\,z)_T&=(T(x+x_1),T(y)\,|\,T(z))=(T(x)+T(x_1),T(y)\,|\,T(z))\\&=(T(x),T(y)\,|\,T(z))+(T(x_1),T(y)\,|\,T(z))\\&=(x,y\,|\,z)_T+(x_1,y\,|\,z)_T\,.\end{aligned}$$

This means that the axioms *ii*)-*v*) of 2-inner product definition are satisfied.

Lemma 3. Let L and L_1 be 2-pre-Hilbert spaces. Then, for the linear mapping $F:L\to L_1$ the following holds true

$$(F(x),F(y)\,|\,F(z))_{L_1}=(x,y\,|\,z)_L\,,\text{ for each } x,y,z\in L \tag{5}$$

if and only if

$$\|F(x),F(y)\|_{L_1}=\|x,y\|_L\,,\text{ for each } x,y\in L\,, \tag{6}$$

and 2-norms on L and L_1 are defined by the 2-inner products.

Proof. Let (5) holds for a linear mapping $F:L\to L_1$. Then for each $x,y\in L$ is true that

$$\|F(x),F(y)\|_{L_1}=(F(x),F(x)\,|\,F(y))_{L_1}=(x,x\,|\,y)_L=\|x,y\|_L\,,$$

i.e. holds (6).

Conversely, if $F:L\to L_1$ is a linear mapping such that holds true (6), then for each $x,y,z\in L$

$$\begin{aligned}(F(x),F(y)\,|\,F(z))_{L_1}&=\tfrac{\|F(x)+F(y),F(z)\|_{L_1}^2-\|F(x)-F(y),F(z)\|_{L_1}^2}{4}\\&=\tfrac{\|F(x+y),F(z)\|_{L_1}^2-\|F(x-y),F(z)\|_{L_1}^2}{4}\\&=\tfrac{\|x+y,z\|_L^2-\|x-y,z\|_L^2}{4}\\&=(x,y\,|\,z)_L\,,\end{aligned}$$

i.e. (5) holds true.

In [6] C. Diminnie and A. White characterized 2-pre-Hilbert space using partial derivatives of 2-functionals, i.e. proved that if $(L,(\cdot,\cdot\,|\,\cdot))$ is a 2-pre-Hilbert space in which the norm is defined by (1), then for each $x,y,z\in L$ holds true

$$(x,y\,|\,z)=\lim_{t\to 0}\tfrac{\|x+ty,z\|-\|x,z\|}{2t}\,.$$

Further, the following Theorem holds true.

Theorem 1 ([4]). Let $(L,\|\cdot,\cdot\|)$ be a 2-norm space. L is 2-pre-Hilbert space if and only if for each $z\in L\setminus\{0\}$ one of the following conditions holds true:

II_1. For each $x,y\in L$ such that $\|x,z\|=\|y,z\|$ and for each $m,n\in\mathbf{R}$ holds true

$$\|mx+ny,z\|=\|nx+my,z\|\,.$$

II_2. $\|x+y,z\|=\|x-y,z\|$, $x,y\in L$ implies

$$\|x+y,z\|^2=\|x,z\|^2+\|y,z\|^2$$

II_3. There is a real number $\alpha\neq 0,\pm 1$ such that

$$\| x,z \| = \| y,z \|,\ x,y\in L \text{ implies } \| x+\alpha y,z \| = \| \alpha x+y,z \|.$$

II_4. There is a real number $\alpha \neq 0,\pm 1$ such that

$$\| x+y,z \| = \| x-y,z \|,\ x,y\in L \text{ implies } \| x+\alpha y,z \| = \| x-\alpha y,z \|.$$

II_5. $\| x,z \| = \| y,z \|$, $x,y\in L$ implies that for each real number $\alpha>0$ holds true

$$\| \alpha x+\alpha^{-1}y,z \| \geq \| x+y,z \|.$$

II_6. For each $x_1,x_2,x_3\in L$ such that $\sum_{i=1}^{3} x_i=0$ and $\| x_1,z \| = \| x_2,z \|$ holds true

$$\| x_1-x_3,z \| = \| x_2-x_3,z \|.$$

II_7. For each $x_1,x_2,x_3,x_4\in L$ such that $\sum_{i=1}^{4} x_i=0$ and $\| x_1,z \| = \| x_2,z \|$ and $\| x_3,z \| = \| x_4,z \|$ holds true

$$\| x_1-x_3,z \| = \| x_2-x_4,z \| \text{ and } \| x_2-x_3,z \| = \| x_1-x_4,z \|.$$

II_8. The value of the expression

$$F(x_1,x_2,x_3)=\| x_1+x_2+x_3,z \|^2+\| x_1+x_2-x_3,z \|^2-\| x_1-x_2-x_3,z \|^2-\| x_1-x_2+x_3,z \|^2$$

does not depend on x_3.

II_9. For each $x_1,...,x_n\in L$, $n\geq 3$ such that $\sum_{i=1}^{n} x_i=0$ the following holds true

$$\sum_{i,k=1}^{n} \| x_i-x_k,z \|^2 = 2n\sum_{i=1}^{n} \| x_i,z \|^2.$$

2. DUNKL-WILLIAMS INEQUALITY INTO 2-NORM SPACE

In this section we will generalize the Dunkl-Williams inequality into 2-normed space. Actually, this inequality was proven in [2], but in our further consideration we will present its proof, and also we will present a proof of the generalization of Mercer inequality ([16]) into 2-normed space.

Theorem 2. a) (Dunkl-Williams inequality). Let L be 2-normed space. Then,

$$\| \tfrac{x}{\|x,z\|}-\tfrac{y}{\|y,z\|},z \| \leq \tfrac{4\|x-y,z\|}{\|x,z\|+\|y,z\|}, \tag{7}$$

for each $z\in L\setminus\{0\}$ and for each $x,y\in L\setminus V(z)$, where $V(z)$ be the subspace generated by the vector z.

b) (Mercer inequality). If L is a 2-pre-Hilbert space, then

$$\| \tfrac{x}{\|x,z\|}-\tfrac{y}{\|y,z\|},z \| \leq \tfrac{2\|x-y,z\|}{\|x,z\|+\|y,z\|}, \tag{8}$$

for each $z\in L\setminus\{0\}$ and for each $x,y\in L\setminus V(z)$, where $V(z)$ is the subspace generated by the vector z.

Proof. a) Let L be 2-normed space, $z\in L\setminus\{0\}$ and $x,y\in L\setminus V(z)$. Then,

$$\begin{aligned}\| x,z \|\cdot\| \tfrac{x}{\|x,z\|}-\tfrac{y}{\|y,z\|},z \| &\leq \| x,z \|\cdot\| \tfrac{x}{\|x,z\|}-\tfrac{y}{\|x,z\|},z \| + \| x,z \|\cdot\| \tfrac{y}{\|x,z\|}-\tfrac{y}{\|y,z\|},z \| \\ &= \| x-y,z \| + |\, \| y,z \| - \| x,z \| \,| \\ &\leq 2\| x-y,z \|.\end{aligned}$$

Analogously one can prove that

$$\| y,z \| \cdot \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \| \le 2 \| x-y,z \|.$$

Finally, by adding the last two inequalities, we get the inequality (7).

b) Let L be 2-pre-Hilbert space, $z \in L \setminus \{0\}$ and $x, y \in L \setminus V(z)$. Then,

$$\begin{aligned}\| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \|^2 &= (\tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|} \mid z) \\ &= 2 - 2(\tfrac{x}{\|x,z\|}, \tfrac{y}{\|y,z\|} \mid z) \\ &= \tfrac{1}{\|x,z\| \cdot \|y,z\|}(2\|x,z\| \cdot \|y,z\| - 2(x,y \mid z)) \\ &= \tfrac{1}{\|x,z\| \cdot \|y,z\|}(2\|x,z\| \cdot \|y,z\| - (\|x,z\|^2 + \|y,z\|^2 - \|x-y,z\|^2)) \\ &= \tfrac{1}{\|x,z\| \cdot \|y,z\|}(\|x-y,z\|^2 - (\|x,z\| - \|y,z\|)^2).\end{aligned}$$

Hence, the above equality and the parallelepiped inequality imply the following

$$\begin{aligned}&\| x-y,z \|^2 - (\tfrac{\|x,z\|+\|y,z\|}{2})^2 \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|}, z \|^2 = \\ &= \| x-y,z \|^2 - (\tfrac{\|x,z\|+\|y,z\|}{2})^2 (\tfrac{1}{\|x,z\| \cdot \|y,z\|}(\|x-y,z\|^2 - (\|x,z\| - \|y,z\|)^2)) \\ &= \tfrac{(\|x,z\|-\|y,z\|)^2}{\|x,z\| \cdot \|y,z\|}(\tfrac{\|x,z\|+\|y,z\|}{2})^2 + \|x-y,z\|^2 \, (1 - (\tfrac{\|x,z\|+\|y,z\|}{2})^2 \tfrac{1}{\|x,z\| \cdot \|y,z\|}) \\ &= \tfrac{(\|x,z\|-\|y,z\|)^2}{4\|x,z\| \cdot \|y,z\|}((\|x,z\| + \|y,z\|)^2 - \|x-y,z\|^2) \ge 0.\end{aligned}$$

Therefore, the inequality (8) holds true.

Theorem 3. Let L be a 2-normed space. The following statements are equivalent:

1) For each $z \in L \setminus \{0\}$ and for each $x, y \in L \setminus V(z)$, where $V(z)$ is a subspace generated by the vector z the inequality (8) holds true.
2) If $x, y, z \in L$ is such that $\| x,z \| = \| y,z \| = 1$, then

$$\| \tfrac{x+y}{2}, z \| \le \| (1-t)x + ty, z \|, \qquad (9)$$

for each $t \in [0,1]$.

Proof. $1) \Rightarrow 2)$. Let suppose that the statement 1) holds true. Let $x, y, z \in L$ be such that $\| x,z \| = \| y,z \| = 1$. Then $z \in L \setminus \{0\}$ and $x, -y \in L \setminus V(z)$. Clearly, for $t = 0$ and $t = 1$, the inequality (9) holds true. If $t \in (0,1)$, then 1) implies the following

$$\begin{aligned}\| (1-t)x + ty, z \| &= (1-t) \| x - \tfrac{t}{t-1} y, z \| \\ &= \tfrac{1-t}{2}(\|x,z\| + \| \tfrac{t}{t-1} y, z \|) \tfrac{2\|x - \frac{t}{t-1} y, z\|}{\|x,z\| + \|\frac{t}{t-1} y, z\|} \\ &\ge \tfrac{1-t}{2}(\|x,z\| + \| \tfrac{t}{t-1} y, z \|) \| \tfrac{x}{\|x,z\|} - \tfrac{\frac{t}{t-1} y}{\|\frac{t}{t-1} y, z\|}, z \| \\ &= \tfrac{1-t}{2}(1 + \tfrac{t}{1-t}) \| x+y,z \| \\ &= \| \tfrac{x+y}{2}, z \|,\end{aligned}$$

i.e. the inequality (9) holds true.

$2) \Rightarrow 1)$. Let suppose that the statement 1) holds true. Let $z \in L \setminus \{0\}$ and $x, y \in L \setminus V(z)$. Then, for $\tfrac{x}{\|x,z\|}, \tfrac{-y}{\|y,z\|} \in L$ holds true $\| \tfrac{x}{\|x,z\|}, z \| = \| \tfrac{-y}{\|y,z\|}, z \| = 1$ and if we let that $t = \tfrac{\|y,z\|}{\|x,z\|+\|y,z\|}$, according to 2) we get that

$$\| \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z \| = 2 \| \frac{\frac{x}{\|x,z\|} + \frac{-y}{\|y,z\|}}{2}, z \|$$

$$\leq 2 \| (1 - \frac{\|y,z\|}{\|x,z\| + \|y,z\|}) \frac{x}{\|x,z\|} + \frac{\|y,z\|}{\|x,z\| + \|y,z\|} \cdot \frac{-y}{\|y,z\|}, z \|$$

$$= \frac{2\|x-y,z\|}{\|x,z\| + \|y,z\|},$$

i.e. the inequality (8) holds true.

Remark 1. In [13] it was proved that for each $x, y, z \in L$ such that the sets $\{x,z\}$ and $\{y,z\}$ are linearly independent the following inequality holds true

$$\| \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z \| \leq \frac{\|x-y,z\| + |\,\|x,z\| - \|y,z\|\,|}{\max\{\|x,z\|, \|y,z\|\}}. \tag{10}$$

Hence, using the fact that for each $x, y, z \in L$ holds true

$$|\,\|x,z\| - \|y,z\|\,| \leq \|x-y,z\|$$

we get that (10) implies the following inequality

$$\| \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z \| \leq \frac{2\|x-y,z\|}{\max\{\|x,z\|, \|y,z\|\}}. \tag{11}$$

Clearly, the inequality (11), which in fact is generalization of Massera and Schäffer inequality ([15]) and holds true into an arbitrary 2-normed space is stronger than Dunkl-Williams inequality (4), but is weaker than the inequality (10).

Also, using the fact that for each $x, y, z \in L$ holds true

$$\|x-y,z\| + |\,\|x,z\| - \|y,z\|\,| \leq \sqrt{2\|x-y,z\|^2 + 2(\|x,z\| - \|y,z\|)^2} \leq 2\|x-y,z\|,$$

the inequality (10) implies that the following inequality holds true

$$\| \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z \| \leq \frac{\sqrt{2\|x-y,z\|^2 + 2(\|x,z\| - \|y,z\|)^2}}{\max\{\|x,z\|, \|y,z\|\}}. \tag{12}$$

Clearly, the inequality (12) is stronger than (11), but weaker than (10).

3. CHARACTERIZATIONS OF 2-PRE-HILBERT SPACE

In this section we will give two characterizations of 2-inner product, which in fact are generalizations of Kirk-Smiley characterization ([10]) and Gurarii-Sozonov characterization ([9]).

Theorem 4. Let L be a 2-normed space. If the following inequality holds true

$$\| \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z \| \leq \frac{2\|x-y,z\|}{\|x,z\| + \|y,z\|}, \tag{13}$$

for each $z \in L \setminus \{0\}$ and for each $x, y \in L \setminus V(z)$, then L be a 2-pre-Hilbert space.

Proof. Let $\alpha > 0$, $z \in L \setminus \{0\}$ and $x, y \in L \setminus V(z)$ be such that $\|x,z\| = \|y,z\|$. The inequality (13), applied to the vectors αx and $-\alpha^{-1} y$ implies the following

$$\|\alpha x + \alpha^{-1} y, z\| \geq \frac{\|\alpha x, z\| + \|\alpha^{-1} y, z\|}{2} \| \frac{\alpha x}{\|\alpha x, z\|} + \frac{\alpha^{-1} y}{\|\alpha^{-1} y, z\|}, z \|$$

$$= \frac{\alpha\|x,z\| + \alpha^{-1}\|y,z\|}{2} \| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \|$$

$$= \frac{\alpha + \alpha^{-1}}{2} \|x+y,z\|$$

$$\geq \|x+y,z\|.$$

This, according to Theorem 1 means that L is a 2-pre-Hilbert space.

Corollary 1. Let L be a 2-normed space. If $x,y,z\in L$ be such that $\|x,z\|=\|y,z\|=1$ holds true and for each $t\in[0,1]$ holds

$$\|\tfrac{x+y}{2},z\|\le\|(1-t)x+ty,z\|,$$

then L is a 2-pre-Hilbert space.

Proof. The proof is a direct implication of Theorem 3 and Theorem 4.

Before we go on the following characterization of 2-pre-Hilbert space, we must mention that the condition II_4 of Theorem 1 is equivalent to the following conditions:

$II_4^{'}$. It exists a real number $\alpha_0>1$ such that $\|\alpha_0 x+y,z\|=\|x+\alpha_0 y,z\|$, for each $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$.

$II_4^{''}$. It exists a real number $t_0\in(0,\frac{1}{2})$ such that

$$\|(1-t_0)x+t_0y,z\|=\|t_0x+(1-t_0)y,z\|$$

for each $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$.

Theorem 1 and the stated above imply the validity of the following Corollary.

Corollary 2. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. L be a 2-pre-Hilbert space if and only if for each $z\in L\setminus\{0\}$ is satisfied one of the following conditions:

1) It exists a real number $\alpha_0>1$ such that $\|\alpha_0 x+y,z\|=\|x+\alpha_0 y,z\|$, for each $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$.
2) It exists a real number $t_0\in(0,\frac{1}{2})$ such that
$$\|(1-t_0)x+t_0y,z\|=\|t_0x+(1-t_0)y,z\|$$
for each $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$.

In the following Theorem, which actually is generalization of Tanaka result ([17]), we will prove that by weakening the conditions 1) and 2) given in Corollary 2, we get a new characterization of 2-pre-Hilbert space.

Theorem 5. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. L is a 2-pre-Hilbert space if and only if for each $z\in L\setminus\{0\}$ is satisfied one of the following conditions:

1) For each $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$ it exists a real number $\alpha>1$ such that $\|\alpha x+y,z\|=\|x+\alpha y,z\|$.
2) For each $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$ it exists a real number $t\in(0,\frac{1}{2})$ such that $\|(1-t)x+ty,z\|=\|tx+(1-t)y,z\|$.

Obviously, the real numbers $\alpha>1$ and $t\in(0,\frac{1}{2})$ can depend on $x,y\in L$, for which $\|x,z\|=\|y,z\|=1$ holds true.

Proof. Corollary 2 implies that it is sufficient to prove that the condition 2) implies that L is a 2-pre-Hilbert space, which according to Corollary 1 means that it is sufficient to prove that the condition 2) implies that $x,y,z\in L$ is such that $\|x,z\|=\|y,z\|=1$, then

$$\|\tfrac{x+y}{2},z\|\le\|(1-t)x+ty,z\|,$$

for each $t\in[0,1]$.

Let $x,y,z\in L$ be such that $\|x,z\|=\|y,z\|=1$. We may assume that the set $\{x,y\}$ is linearly independent. Consider the set

$$A=\{t\in(0,\tfrac{1}{2})\mid \|(1-t)x+ty,z\|=\|tx+(1-t)y,z\|\}.$$

The condition 2) implies that $A\neq\varnothing$. Therefore it exists $t_0=\sup A$. We will prove that $t_0=\frac{1}{2}$. Since the convexity of the function $t\to\|(1-t)x+ty,z\|$, $t\in[0,1]$, we will get that

$$\|\tfrac{x+y}{2},z\|\leq\|(1-t)x+ty,z\|,$$

for each $t\in[0,1]$.

Let suppose that $t_0<\frac{1}{2}$. Then, the continuous of 2-norm implies that $t_0\in A$. The vectors $u=(1-t_0)x+t_0y$ and $v=t_0x+(1-t_0)y$ satisfy $\|u,z\|=\|v,z\|$. Let be $x_0=\frac{u}{\|u,z\|}$ and $y_0=\frac{v}{\|v,z\|}$. The assumption implies that it exists a real number $t_1\in(0,\frac{1}{2})$ such that

$$\|(1-t_1)x_0+t_1y_0,z\|=\|t_1x_0+(1-t_1)y_0,z\|.$$

Let $t^*=(1-t_1)t_0+t_1(1-t_0)$. Then $t_0<t^*<\frac{1}{2}$ and also holds

$$\|(1-t^*)x+t^*y,z\|=\|t^*x+(1-t^*)y,z\|.$$

This means that $t^*\in A$, and that is contradictory to $t_0=\sup A$ and $t_0<t^*<\frac{1}{2}$. Finally, the contradictory implies that $t_0=\frac{1}{2}$.

Corollary 3. Let $(L,\|\cdot,\cdot\|)$ be a 2-normed space. L be a 2-pre-Hilbert space if and only if for each $z\in L\setminus\{0\}$ is satisfied that

$$\|x+\tfrac{x+y}{\|x+y,z\|},z\|=\|y+\tfrac{x+y}{\|x+y,z\|},z\|, \tag{14}$$

for each $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$ and $x+y\notin V(z)$.

Proof. We will prove that the condition is sufficient. If $z\in L\setminus\{0\}$ and $x,y\in L$ be such that $\|x,z\|=\|y,z\|=1$ and $x+y\notin V(z)$, then

$$\begin{aligned}\|(1+\|x+y,z\|)x+y,z\|^2&=(1+\|x+y,z\|)^2\|x,z\|^2+2(1+\|x+y,z\|)(x,y\mid z)+\|y,z\|^2\\&=(1+\|x+y,z\|)^2\|y,z\|^2+2(1+\|x+y,z\|)(x,y\mid z)+\|x,z\|^2\\&=\|x+(1+\|x+y,z\|)y,z\|^2,\end{aligned}$$

i.e. the following equality holds true

$$\|(1+\|x+y,z\|)x+y,z\|=\|x+(1+\|x+y,z\|)y,z\|, \tag{15}$$

which is equivalent to the equality (14).

We will prove that the condition is necessary. Let $z\in L\setminus\{0\}$ and $x,y\in L$ be such that $\|x,z\|=\|y,z\|=1$ and $x+y\notin V(z)$. Then holds true the equality (14), which is equivalent to (15). But, $x+y\notin V(z)$, and therefore $1+\|x+y,z\|>1$. The last, according to Theorem 5, means that L is a 2-pre-Hilbert space.

Remark 2. Since $1+\|x+y,z\|$ depends on $x,y\in L$ such that $\|x,z\|=\|y,z\|=1$, we can deduce that the statement given in Corollary 2 is not an implication of the statements given in Corollary 1. The last actually shows the advantage of Theorem 5.

Example 1. In [11] it is proved that in the set of bounded arrays of real numbers l^∞ by

$$\|x,y\|=\sup_{\substack{i,j\in\mathbf{N}\\ i<j}}\left|\begin{vmatrix}x_i & x_j\\ y_i & y_j\end{vmatrix}\right|,\ x=(x_i)_{i=1}^\infty,\ y=(y_i)_{i=1}^\infty\in l^\infty$$

is defined a 2-norm. The last means that $(l^{\infty},\|\cdot,\cdot\|)$ is real 2-normed space. It is easy to find that the vectors

$$x=(1-\tfrac{1}{2},1-\tfrac{1}{2^2},...,1-\tfrac{1}{2^n},...),\ y=(0,\tfrac{1}{2}-1,\tfrac{1}{2^2}-1,...,\tfrac{1}{2^{n-1}}-1,...) \text{ and } z=(1,0,0,...,0,...)$$

satisfy followings $\|x,z\|=\|y,z\|=1$ and $x+y\notin V(z)$. Further, $\|x+y,z\|=\frac{1}{2^2}$, and therefore

$$x+\tfrac{x+y}{\|x+y,z\|}=(1+\tfrac{3}{2},1+\tfrac{3}{2^2},1+\tfrac{3}{2^3},...,1+\tfrac{3}{2^n},...),\ y+\tfrac{x+y}{\|x+y,z\|}=(2,\tfrac{1}{2},\tfrac{3}{4}-1,...,\tfrac{3}{2^{n-1}}-1,...).$$

So,

$$\|x+\tfrac{x+y}{\|x+y,z\|},z\|=\tfrac{7}{4}\neq 1=\|y+\tfrac{x+y}{\|x+y,z\|},z\|.$$

The last according to corollary 3, means that the 2-normed space $(l^{\infty},\|\cdot,\cdot\|)$ is not 2-pre-Hilbert space.

4. Conclusion

In example 1, by using Corollary 3 is proven that $(l^{\infty},\|\cdot,\cdot\|)$ is not 2-pre-Hilbert space. Analogously, other results obtained in this paper may find application in checking whether a 2-normed space is 2-pre-Hilbert, as is the case with spaces $(L^p(\mu),\|\cdot,\cdot\|),\ p>1$.

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