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Cohen's Theorem for Power and Laurent Series Rings

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Abstract: *This paper is an investigation on the concepts of prime left ideal and left Noetherian ring. In particular it is proved that the well known Cohen's theorem holds for left Noetherian rings, it has been given an easy proof for Power series and Laurent series rings [[R[X,α]], where X is an indeterminate over R, α is a surjective endomorphism of the ring R.*

Keywords: *Commutative ring, Noetherian ring, Prime ideal, Power Series ring, Surjective endomorphism of a ring.*

1. INTRODUCTION

The concept of prime right ideals and right Noetherian rings was intended by Gerhard O Michler [1971]. We extend the concept of Cohen's theorem in Left Noetherian Rings. Using the characterizations of prime left ideals and left Noetherian rings, it has been given an easy proof for Power series and Laurent series ring [[R[X,α]], where X is an indeterminate over R, α is a surjective endomorphism of the ring R.

2. PRELIMINARIES

2.1 Cohen Theorem

[Cohen, I, S. [1950]**]:** R is Noetherian if and only if every prime ideal of R is finitely generated.

2.2 Note

Using the following definition [1.3] of left prime ideal we show in this note that Cohen's theorem holds for left Noetherian rings.

2.3 Definition

[Lambek,J.[1966]**]:** The left ideal A of the ring R is prime if tRs $\nsubseteq$ A whenever t and s do not belong to A. (that is tRs is not contained in A).

2.4 Note:

Equivalently Let P be a prime ideal in a ring R with unity. We can show that the following are equivalent.

(i) If a, b $\in$ R are such that aRb $\subseteq$ P, then a $\in$ P or b $\in$ P.

(ii) If a, b $\in$ R are such that RaRb (respectively aRbR) $\subseteq$ P, then a $\in$ P or b $\in$ P.

(iii) If I and J are left (respectively right) ideals in R such that IJ$\subseteq$P, then either I $\subseteq$ P or J $\subseteq$ P.

2.5 Note:

As an application of our result we give a proof for the fact that the Power series ring, Laurent series ring $R[[X,\alpha]]$ is left Noetherian if R is left Noetherian.

2.6 Ring of Power Series [Musili, C. [1994]]:

Given a ring R let R[[x]] be the formal power series with coefficient in R where R[[x]] = {$a_0 + a_1x + a_2x^2 + \dots \dots \dots a_ix^i \in R$}

2.7 Unitary Module [Musili, C. [1994]]:

A left R-module M is said to be unitary left R-module if 1x = x ∀ x ∈ M

2.8 Power Series Ring in Several Variables [Musili, C[1994]]:

We can construct power series rings in two variables X and Y over a given ring R, which by definition is (R[X])[[Y]] and is denoted by R[[X,Y]]. We have R[[X,Y]] = $\sum_{\substack{i+j=0 \\ i,j\geq 0}}^{\infty} a_{ij} X^i Y^j / a_{ij} \in R$.

Again if X_1, X_2, … … X_n are finitely many indeterminates, we have the power series ring in n variables R[[X_1, X_2, … … X_n]].

2.9 Left Noetherian [Macdonald, I.G, and Atiyah, M.F.[1969]]:

A ring R is called left Noetherian if it is a Noetherian as a left module over itself. i.e., ascending chain condition or maximal condition holds for left ideals of R.

2.10 Noetherian Ring [Macdonald, I.G, and Atiyah, M.F.[1969]]:

(i)An A-module M is said to be Notherian if the poset of sub-modules of M satisfies ascending chain condition. That is every ascending chain of sub-modules of M becomes stationary.

(ii)A-module M is Notherian if and only if every non-empty family of sub-modules of M contains a maximal element..

2.11 Laurent Series [Musili, C. [1994]]:

Given a ring R, a formal expression of the form $a_{-n} X^{-n} + a_{-n+1}X^{-n+1} + \ldots + a_{-1}X^{-1} + a_0 + a_1X + \ldots$ where $a_i \in R$ and n ∈ Z^+, is called a Laurent series in the variable X with coefficients in R. In this expression $a_{-n}X^{-n} + a_{-n+1}X^{-n+1} + \ldots + a_{-1}X^{-1}$ is called the principal part and $a_0 + a_1X + \ldots$ is called the power series part. The set of all Laurent series is denoted by R <X>, we have R<X> = $\{a_{-n}X^{-n} + \ldots + a_{-1}X^{-1} + a_0 + a_1X + \ldots \mid a_i\in R, n \in Z^+\}$

Equality, addition and multiplication for the Laurent series follow the same rule as in the power series. For example, product of two Laurent series is given by

$$\left(\sum_{\substack{n=-m \\ m\geq 0\ fixed}}^{\infty} a_n X^r\right)\left(\sum_{\substack{r=-l \\ l\geq 0\ fixed}}^{\infty} b_r X^r\right) = \sum_{s=-(m+1)}^{\infty} c_s X^s$$

Where

$$c_s = \sum_{n,r}^{n+r=s} a_n b_r$$

2.12 Finitely Generated Ideal [Musili, C. [1994]]:

An ideal (left/right/2-sided) is said to be finitely generated if there is a finite subset X = {x_1, … , x_n)of Isuch that the ideal generated by X is I (i.e., every element of I can be expressed as a (left/right/2-sided) R-linear combination of (x_1, …, x_n).

2.13 Left Ideal Generated by X[Musili, C. [1994]]:

Let X be a subset of a ring R. The left ideal generated by X is the smallest left ideal in R containing X, or equivalently, it can be defined as the intersection of all the left ideals in R containing X which can be seen to be equal to

$$\left(\sum_{\substack{n_i\in Z \\ x_i\in X}}^{finite} n_i\, x_i + \sum_{\substack{r_j\in R \\ y_i\in X}}^{finite} y_j\, r_j\right)$$

In particular note that

(i) If $X = \phi$, the left ideal generated by ϕ is (0).

(ii) If $X = \{x\}$, the left ideal generated by x is $\{nx + xr \mid n \in Z, r \in R\}$,

(=$\{xs \mid s \in R\}$, if $1 \in R$). This is denoted by $(x)_r$ (left ideal generated by x) and is called the principal left ideal generated by x.

2.14 Finitely Generated Module [Musili, C. [1994]]:

An R-module M is said to be finitely generated over R if there is a finite subset X of M such that M is the sub module generated by X, i.e., if $X = \{x_i, \dots , x_r)$, then (Assuming R has 1 and M is unitary) we have

$$M = \left\{ \sum_{i=1}^{r} [(a_i x_i | a_i \in R)] \right\}$$

3. Prime Left Ideals and Power and Laurent Series of Left Noetherian Rings

3.1 Note:

In this paper all rings have an identity element and has associative ring homomorphism and modules are unitary. The left ideal A of the ring R is finitely generated as a left R-module.

3.2 Lemma:

If A is finitely generated left ideal and B is finitely generated two sided ideal then AB is finitely generated.

Proof: If A $= \sum_{i=1}^{n} Ra_i \qquad [Left - ideal]$

$= Ra_1 + Ra_2 + Ra_3 + \dots \dots \dots \dots$

$= r_1a_1 + r_2a_2 + r_3a_3 + \dots \dots \dots + r_na_n / r_i \in R$

B $= \sum_{j=1}^{m} Rb_j$

$= Rb_1 + Rb_2 + Rb_3 + \dots \dots \dots \dots$

$= r_1b_1 + r_2b_2 + r_3b_3 + \dots \dots \dots + r_mb_m / r_j \in R$

$$AB = \sum_{i=1}^{n} Ra_i \sum_{j=1}^{m} Rb_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} Ra_i Rb_j$$

$$= \sum_{i=1}^{n} a_i (r_1 b_1 + r_2 b_2 + \dots\dots + r_m b_m / r_i \in R)$$

$$= \sum_{i=1}^{n} a_i (r_i b_i)$$

$= a_1r_1b_1 + a_2r_2b_2 + \dots \dots \dots$

Hence AB is finitely generated

3.3 Lemma:

If $P_1, P_2 \dots P_k$ are finitely generated left ideals of the ring R such that R/P_i is left Noetherian for i = 1,2,……k. Then every left ideal A of R satisfying $P_1P_2 \dots\dots P_k \le A \le \bigcap_{i=1}^{k} P_i$ is finitely generated.

Proof: By lemma 3.2 the decending chain $P_1 \ge P_1P_2 \ge \dots P_1P_2 \dots P_{k-1}P_k \ge P_1P_2 \dots P_{k-1}P_k$ contains finitely generated left ideals of R.

Since $P_1P_2 \dots \dots P_i/P_1P_2 \dots \dots P_1\ P_{i+1}$ is a finitely generated R/P_{i+1} module over the left Noetherian ring R/P_{i+1} for every i = 1,2,3……… k-1, it follows that every sub module of the left R-module $P_1/P_1P_2 \dots\dots P_k$ is finitely generated. Thus $A/P_1P_2\dots\dots\dots P_k$ is finitely generated. Therefore the left ideal A of R is finitely generated.

3.4 Lemma:

If every prime left ideal of R is a finitely generated left ideal of R, then R is left Noetherian.

Proof: Since every left ideal of R is finitely generated, R satisfies the ascending chain condition on left ideals.

Thus if R is not left Noetherian, Then there is an ideal M of R which is maximal among the ideals X of R such that R/X is not left Noetherian. We may assume that M=0, and that the set T of left ideals Y of R which are not finitely generated is not empty. (i.e., T = {The set of left ideals Y of R} is not finitely generated.

Thus $T \neq \emptyset$

By Zorn's lemma, T has a maximal element G.

Since G is not finitely generated, G is not prime ideal.

Thus there are $a \notin G$ and $b \notin G$ such that $bRa \leq G$.

Clearly A = Ra + G is finitely generated

Since U = RbR is a finitely generated left ideal of R, AU is a finitely generated left ideal of R contained in G.

If R =RbR, then $a \in Ra = RaRb \leq G$.

It is a contradiction [since $a \notin G$ but $a \in G$]

Thus R/U is left Noetherian and G/AU is a finitely generated R-module.

Hence G is a finitely generated left ideal of R.

This Shows that R is left Noetherian.

3.5 Lemma:

If every left prime ideal of R is a finitely generated left ideal of R, then every left ideal A of R contains a product of finitely many prime ideals P_i of R each containing S.

Proof: In contrary way suppose that the set R of all left ideals A of R not containing a product of finitely many prime ideals. $A \leq P_i$ of R is not empty.

If $X_1 < X_2 < \dots \dots \dots$ is an infinite properly ascending chain of elements $X_i \in R$, then $X = \bigcup_{i=1}^{\infty} X_i$ also belongs to R, because otherwise these would be finitely many prime ideals, $Q_j \geq X$, j = 1,2,……k such that $Q_1Q_2 \dots \dots \dots Q_k \leq X$, since $Q_1\ Q_2 \dots \dots Q_k$ is a finitely generated left ideal of R by lemma 3.2, it follows that

$T=Q_1\ Q_2 \dots \dots \dots Q_k \leq X_s$ for some integer s.

Hence $X_s \in R$, a contradiction.

Therefore by Zorn's Lemma R has a maximal element G.

Clearly G is not a left prime ideal.

Hence there is an ideals $A_i > G$, i = 1,2, such that $A_1\ A_2 \leq G$.

Since $A_i \notin R$, there are prime ideals P_{ij} of R such that

$$P_{i1}\ P_{i2} \dots \dots \dots P_{ik} \leq A_i \leq \bigcap_{j=1}^{K_i} P_{ij}\ , \quad i = 1,2$$

Thus $P_{11}\, P_{12} \ldots \ldots \ldots P_{1k_1}\, P_{21}\, P_{22} \ldots \ldots \ldots P_{2K_2} \leq A_1A_2 \leq G < \left(\bigcap_{j=1}^{K_1} P_{ij}\right) \cap \left(\bigcap_{j=1}^{K_2} P_2\right)$

By lemma 3.3, G is finitely generated.

This contradiction proves lemma 3.5.

This implies the set R of all left prime ideals A of R contains a product of finitely many primes ideals P_i of R each containing S.

3.6 Lemma:

If every left prime ideal of the ring R is a finitely generated left ideal of R, then R satisfies the ascending chain condition on left prime ideals.

Proof: Let X = $\bigcup_{i=1}^{\infty} P_i$

By Lemma 3.5 there are finitely many prime ideal Q_j, j = 1,2, … … k of R such that

$$T=Q_1Q_2 \ldots \ldots \ldots Q_k \leq X \leq \bigcap_{j=1}^{k} Q_j \,.$$

By lemma 3.2, T is a finitely generated left ideal of R.

Thus $T=Q_1\, Q_2\, Q_3 \ldots \ldots \ldots Q_k \leq P_s$ for some integer S.

Since P_S is a left prime ideal of R, we have $Q_j \leq P_s < X \leq Q_j$ for some j ∈ {1, 2,…… k}

This shows that R satisfies the ascending chain condition on left prime ideals.

3.7 Theorem

The ring R is left Noetherian if and only if every prime left ideal of R is

Finitely generated

Proof: If every left prime ideal of R is finitely generated then by Lemma 3.4 it suffices to show that every left prime ideal of R is a finitely generated left ideal. This follows at once from lemma 3.5 and lemma 3.3.

If R/P is left Noetherian for every left prime ideal P of R.

Let ρ be the set of all prime ideals P of R such that R/P is not left Noetherian.

Since every left prime ideal is a finitely generated left ideal of R,

R satisfies the ascending chain condition on left prime ideal by lemma 3.6.

Thus if ρ is not empty, ρ contains a maximal element M.

We may assume that M = 0.

Since R is not left Noetherian there exists a left prime ideal

A ≠ 0 of R which by lemma 3.4 is not a finitely generated left ideal of R.

By Lemma 3.5, there are finitely many prime ideals P_i>A,

i = 1, 2, … … … K of R such that $P_1\, P_2 \ldots \ldots \ldots P_k \leq A$

Since $P_i \neq 0$ for all i, the rings R/P_i are left Noetherian by the choice of M.

Hence by lemma 3.3 A is a finitely generated left ideal of R, a contradiction.

Thus ρ is empty and R is left Noetherian.

Converse, assume the ring R is left Noetherian then by lemma 3.3, every left prime ideal of R is finitely generated.

3.8 Corollary [Lambek, J.[1966]]

If R is left Noetherian, then R[[x]] is left Noetherian

3.9 Note:

Actually, a more general result holds. Let α be a surjective endomorphism of the ring R, and let X be an indeterminate over R.

As usual R[[x, α]] denotes the ring of Power series.

f(x) = $\sum_{i=0}^{\infty} r_i X^i$, $r_i \in$R. with coefficient wise addition and the following distributively extended multiplication $X_r = \alpha(r)X$ for all r $\in$R

3.10 Corollary

If R is a left Noetherian, then the power series ring S = R[[X, α]] is left Noetherian.

Proof: (cf Kaplansky [1970] p, 48) By theorem 3.7 it suffices to show that every prime left ideal P of S is finitely generated

Let$\emptyset$: S $\longrightarrow$R be the natural ring epimophirm obtained by mapping X onto zero.

Let $\emptyset$ (P) = $\sum_{i=1}^{n} Ra_i$ where $A_i \in \emptyset$ (P).

Then there are power series $f_i \in$ P such that a_i is the constant term of f_i.

If X $\in$ P, then P = $\sum_{i=1}^{n} f_i$ S + XS

Thus we may assume that X $\notin$ P.

If g $\in$ P and g_0 is the constant term of g, then g_0 = $\sum_{i=1}^{n} r_{i0} a_i$ for some $r_{i0} \in$ R

Thus g - $\sum_{i=1}^{n} r_{i0} f_i$ = g_1X for some $g_1 \in$ S

Furthermore, $g_1X \in$ P.

Therefore $g_1X^n \in$ P and $g_1X_r = g_1\alpha(r)X \in$P for all r $\in$ R

Since α is surjective , it follows that $g_1RX \le$ P, thus $g_1 \in$ P,

because X$\in$P and P is left prime ideal of S.

In the same way we obtain g_1 = $\sum_{i=1}^{n} r_{i1} f_i$ + g_2X with $g_2 \in$ P

Thus there are $h_i \in$R[[X, α]], where h_i = $\sum_{i=1}^{\infty} r_{ij} X^j$ such that g = $\sum_{i=1}^{n} h_i f_i$,

Hence P is a finitely generated left ideal of S.

This implies the power series ring S = R[[X, α]] is left Notherian.

3.11 Corollary

If R is a left Noetherian, then the Laurent series ring S = R[[X, α]] is left Noetherian.

Proof: (c.f.Kaplansky, I[1970] P.48) By theorem 3.7 it suffices to show that every prime left ideal P of S is finitely generated.

Let $\emptyset$: S $\longrightarrow$R be the natural ring epimophirm obtained by a mapping X onto zero.

Let $\emptyset$ (P) = $\sum_{i=-m}^{n} Ra_i$ where $A_i \in \emptyset$ (P)

Then there are Laurent series $f_i \in$ P such that a_i is the constant term of f_i.

If X ∈ P, then P = $\sum_{\substack{i=-m \\ m\geq 0}}^{n} f_i S$ + XS

Thus we may assume that X ∉ P.

If g ∈ P and g_0 is the constant term of g, then g_0 = $\sum_{\substack{i=-m \\ m\geq 0\ fixed}}^{n} r_{i0} a_i$ for some r_{i0}∈ R

Thus g - $\sum_{\substack{i=-m \\ m\geq 0\ fixed}}^{n} r_{i0} f_i$ = g_1X for some g_1∈ S

Furthermore, g_1X∈ P.

Therefore g_1X^n∈ P and g_1X_r= $g_1\alpha(r)$X∈P for all r ∈ R

Since α is surjective, it follows that g_1RX ≤ P, thus g_1∈ P,

because X∈P and P is left prime ideal of S.

In the same way we obtain g_1 = $\sum_{i=-m}^{n} r_{i1} f_i$ + g_2X with g_2∈ P

Thus there are h_i∈R[[X, α]], where h_i = $\sum_{\substack{i=-m \\ m\geq 0\ fixed}}^{\infty} X^j r_{ij}$ such that g = $\sum_{i=-m}^{n} h_i f_i$,

Hence P is a finitely generated left ideal of S.
This implies the Laurent series ring S = R[[X, α]] is left Notherian.

4. Conclusion

This work has given a prominence on concepts of prime left ideals, left Noetherian rings and particularly on a simple proof of Cohen's theorem. Using the characterizations of prime left ideals and left Noetherian rings, it has been given an easy proof for Power series and Laurent series ring [[R[X,α]], where X is an indeterminate over R, α is a surjective endomorphism of the ring R.

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