International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 2, February 2015, PP 79-89
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

On Pseudo SCHUR Complements in an EP Matrix

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Abstract: *It is established that under certain conditions a pseudo schur complement in an EP matrix is as well an EP matrix. As an application a decomposition of a partitioned matrix into a sum of EP matrices is given.*

Keywords: *EP matrix, Pseudo schur complements, partitioned matrix.*

1. INTRODUCTION

All matrices considered here are complex matrices and * will indicate the forming of the conjugate transpose matrix. For an $m \times n$ matrix *A*, any matrix *X* satisfying $AXA = A$ is called a generalized inverse of *A* and is denoted by A^{-}. The distinctive notation $A^{\dagger}$ is used for the Moore-Penrose inverse of *A* [1]. A square matrix *A* is said an EP matrix if $N\ A = N\ A^{*}$, where $N\ A$ denotes the null space of *A* and is said on $EP_r^{'}$ matrix if A is EP and $\rho(\mathrm{A}) = r$, where $\rho(\mathrm{A})$ is the rank of A. It is proved in [2] that *A* is EP iff $AA^{\dagger} = A^{\dagger}A$. A well known lemma concerning generalized inverses is the following.

Lemma1.1. *([3], p.21)*

If *X* and *Y* are generalized inverses of *A*, then $CXB = CYB$ if and only if $N(A) \subseteq M(C)$ and $N(A^{*}) \subseteq N(B^{*})$ or, equivalently, if and only if

$$C = CA^{-}A \text{ and } B = AA^{-}B \text{ for every } A^{-} \tag{1}$$

Throughout this paper we are concerned with $n \times n$ matrices *M* partitioned in the form

$$M = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{2}$$

Where *A* and *D* are square matrices. With respect to this partitioning a Pseduo Schur complement of A_{11} in *M* is a matrix of the form $M/A_{11} = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} A_{11}^{\dagger}\ A_{12} \quad A_{13}$.

For properties of Pseduo Schur complements one may refer to [4], [5] and [6]. On account of Lemma 1.1 it is obvious that under certain conditions M/A_{11} is independent of the choice of $A^{\dagger}$. However in the sequel we shall always assume that M/A_{11} is given in terms of specific choice of $A^{\dagger}$.

In [7] necessary and sufficient conditions are derived for a matrix of the form (2) with $A_{12} \quad A_{13} = 0$ (or $\begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = 0$) to be EP. The results are here extended for general matrices of the form (2). If a partitioned matrix of the form (2) is EP, then in general M/A_{11} is not EP. Here we determine necessary and sufficient conditions for M/A_{11} to be EP. In particular, when $\rho(M)=\rho(A_{11})$ our result include as special cases the result of paper [8]. In [6] we have given here a decomposition of a partitioned matrix into a sum of EP matrices. Further it is shown that in an EP_r matrix every principal every principal submatrix of rank *r* is EP_r.

The motivation for our research is the following:

1. The paper of A.R. Meenakshi [6] in which she extended the result of Katz, I.J and Pearl, M.H., [9] considering the conditions of EP_r matrices, normal EP_r matrices and sums of EP_r matrices.
2. The paper of D.Carlson, E.Haynsworth and T.H.Markhasm [10] in which they gave detailed explanation of the concept generalization of the schur complements by means of the Moore-Penrose inverse.
3. The paper of Drazin, M.P in which he had explained the concept of pseudo-inverse in associate rings and semi groups.

Our purpose is to generalize these result and aspects for the result of pseudo schur complement of EP matrix of order 3x3.

2. PSEUDO SCHUR COMPLEMENT MATRIX

$$K=\begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \qquad B=\begin{bmatrix} K/N_{11} & K/N_{12} & K/N_{13} \\ K/N_{21} & K/N_{22} & K/N_{23} \\ K/N_{31} & K/N_{32} & K/N_{33} \end{bmatrix}$$

$$BKN=\begin{bmatrix} B/K/N_{11} & B/K/N_{12} & B/K/N_{13} \\ B/K/N_{21} & B/K/N_{22} & B/K/N_{23} \\ B/K/N_{31} & B/K/N_{32} & B/K/N_{33} \end{bmatrix} \tag{3}$$

$$\left[BKN/\ B/K/N_{11}\right]=\begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix}-\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12} \quad B/K/N_{13}\ \right]$$

3. RESULTS

Theorem3.1

Let BKN be matrix of the form (3) with $N\ B/K/N_{11} \subseteq \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$ and $N\left[BKN/\ B/K/N_{11}\right] \subseteq N\left[\ B/K/N_{12} \quad B/K/N_{13}\ \right]$, then the following are equivalent.

(i) BKN is an EP matrix.

(ii) $B/K/N_{11}$ and $\left[BKN/\ B/K/N_{11}\right]$ are EP, $N\ B/K/N_{11}{}^{*} \subseteq \left[\ B/K/N_{12} \quad B/K/N_{13}\ \right]$ and $N\left[BKN/\ B/K/N_{11}\right]^{*} \subseteq M\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$;

(iii) Both the matrices $\begin{bmatrix} B/K/N_{11} & 0 \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} & \left[BKN/\ B/K/N_{11}\right] \end{bmatrix}$ and $\begin{bmatrix} B/K/N_{11} & \left[\ B/K/N_{12} \quad B/K/N_{13}\ \right] \\ 0 & \left[BKN/\ B/K/N_{11}\right] \end{bmatrix}$ are EP.

Proof

(i) $\Rightarrow$ (ii). Let us consider the matrices $P=\begin{bmatrix} I & 0 \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}(B/K/N_{11})^{\dagger} & I \end{bmatrix}$,

$Q=\begin{bmatrix} I & \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}\begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix}^{\dagger} \\ 0 & I \end{bmatrix}$ and $L=\begin{bmatrix} B/K/N_{11} & 0 \\ 0 & \begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix} \end{bmatrix}$. Clearly P and Q are nonsingular. By assumption $N(B/K/N_{11}) \subseteq N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$ and

$N\begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix} \subseteq N\begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}$ by using lemma 1.1 it is obvious that BKN can be factorized as $BKN = PQL$. Hence $\rho(BKN)=\rho(L)$ and $N(BKN)=N(L)$. But BKN is EP. E.g., $N(BKN)^{*}=N(BKN)=N(L)$. Therefore by using lemma 1.1 again

$(BKN)^{*}=(BKN)^{*}L^{\dagger}L$ holds for every $L^{\dagger}$ one choice of $L^{\dagger}$ is

$L^{\dagger}=\begin{bmatrix} (BKN)^{\dagger} & 0 \\ 0 & \begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix}^{\dagger} \end{bmatrix}$, which gives

$$(BKN)^{*}=\begin{bmatrix} (B/K/N_{11})^{*} & \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^{*} \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}^{*} & \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix}^{*} \end{bmatrix}$$

$$\begin{bmatrix} (B/K/N_{11})^{\dagger}(B/K/N_{12}) & 0 \\ 0 & \begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix}^{\dagger}\begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix} \end{bmatrix}$$

$(BKN)^{*}=(BKN)^{*}(BKN)^{\dagger}(BKN)$ implies $N(BKN)^{*}\supseteq N(BKN)$, and since "$(BKN)^{*}=$" BKN these imply $N(BKN)^{*}=N(BKN)$. Hence BKN is EP. From

$\begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^{*}=\begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^{*}(B/K/N_{11})^{\dagger}(B/K/N_{11})$ it follows that

$N\begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^{*}\supseteq N(B/K/N_{12})=N(B/K/N_{12})^{*}$.

After substituting

$\begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix}$

$=\begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix}+\begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}(B/K/N_{11})^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$ and using

$\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}^{*}=\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}^{*}\begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix}^{\dagger}\begin{bmatrix} BKN/(B/K/N_{11}) \end{bmatrix}$ in

$$\begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix}^{*} = \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix}^{*} \left[BKN/\ B/K/N_{11}\right]^{\dagger} \left[BKN/\ B/K/N_{11}\right]$$

we get

$$\left[BKN/\ B/K/N_{11}\right]^{*} = \left[BKN/\ B/K/N_{11}\right]^{*} \left[BKN/\ B/K/N_{11}\right]^{\dagger} \left[BKN/\ B/K/N_{11}\right].$$

This implies $N\left[BKN/\ B/K/N_{11}\right]^{*} \supseteq N\left[BKN/\ B/K/N_{11}\right]$ and since

$\rho\left[BKN/\ B/K/N_{11}\right]^{*} = \rho\left[BKN/\ B/K/N_{11}\right]$ we get $N\left[BKN/\ B/K/N_{11}\right]^{*} = N\left[BKN/\ B/K/N_{11}\right]$.

Thus $\left[BKN/\ B/K/N_{11}\right]$ is EP, further

$$N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} \supseteq N\left[BKN/\ B/K/N_{11}\right] = N\left[BKN/\ B/K/N_{11}\right]^{*}.$$

Hence (ii) holds.

(ii) $\Rightarrow$ (i). Since $N\ B/K/N_{21} \subseteq N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$ $N\ B/K/N_{21}{}^{*} \subseteq N\left[\ B/K/N_{21}\ \ B/K/N_{21}\right]^{*}$

$N\left[BKN/\ B/K/N_{11}\right] \subseteq N\left[\ B/K/N_{12}\ \ B/K/N_{13}\right]$ and $N\left[BKN/\ B/K/N_{11}\right]^{*} \subseteq N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}^{*}$

holds according to the assumption, it can be applied (v) of Theorem 1 of the paper [4] and so $BKN^{\dagger}$ is given by the formula

$$BKN^{\dagger} = \begin{bmatrix} B/K/N_{11}{}^{\dagger} + B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{11}\ \ B/K/N_{11}\right] & -\ B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\ \ B/K/N_{13}\right] \\ \left[BKN/\ B/K/N_{11}\right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger} & \left[BKN/\ B/K/N_{11}\right]^{\dagger} \\ -\left[BKN/\ B/K/N_{11}\right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger} & \left[BKN/\ B/K/N_{11}\right] \end{bmatrix} \quad (4)$$

According to lemma 1.1 the assumptions $N\ B/K/N_{11} \subseteq N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$ and

$N\ B/K/N_{11}{}^{*} \subseteq N\ B/K/N_{11}{}^{*} \subseteq N\left[\ B/K/N_{12}\ \ B/K/N_{13}\right]^{*}$ imply that

$\left[BKN/\ B/K/N_{11}\right]$ is invariant for every choice of $B/K/N_{11}{}^{\dagger}$.

Hence

$$\left[BKN/\ B/K/N_{11}\right] = \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix} - \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\ \ B/K/N_{13}\right]$$

further, using $\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} = \left[BKN/\ B/K/N_{11}\right]\left[BKN/\ B/K/N_{11}\right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$ and

$\left[\ B/K/N_{12}\ \ B/K/N_{13}\right] = \ B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\ \ B/K/N_{13}\right]$, $BKN\ \ BKN^{\dagger}$ is reduced to the form

$$(BKN)(BKN)^{\dagger}=\begin{bmatrix}(B/K/N_{11})(B/K/N_{11})^{\dagger} & 0\\ 0 & [BKN/(B/K/N_{11})][BKN/(B/K/N_{11})]^{\dagger}\end{bmatrix}$$ Using

$$[(B/K/N_{12})\ (B/K/N_{13})]=[(B/K/N_{12})\ (B/K/N_{13})][BKN/(B/K/N_{11})]^{\dagger}[BKN/(B/K/N_{11})]$$

and $\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}=\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}(B/K/N_{11})^{\dagger}(B/K/N_{11})$, a similar way gives

$$(BKN)(BKN)^{\dagger}=\begin{bmatrix}(B/K/N_{11})^{\dagger}(B/K/N_{11}) & 0\\ 0 & (B/K/N_{11})(B/K/N_{11})^{\dagger}\end{bmatrix}.$$

The relations $(B/K/N_{11})(B/K/N_{11})^{\dagger}=(B/K/N_{11})^{\dagger}(B/K/N_{11})$ and

$[BKN/(B/K/N_{11})][BKN/(B/K/N_{11})]^{\dagger}=[BKN/(B/K/N_{11})]^{\dagger}[BKN/(B/K/N_{11})]$ results $(BKN)(BKN)^{\dagger}=(BKN)^{\dagger}(BKN)$, (BKN) is EP. Thus (i) holds (ii)$\Leftrightarrow$(iii). By corollary 8 in [7]

$\begin{bmatrix}B/K/N_{11} & 0\\ \begin{bmatrix}B/K/N_{21}\\ B/K/N_{32}\end{bmatrix} & [BKN/(B/K/N_{11})]\end{bmatrix}$ is EP iff $(B/K/N_{11})$ and $[BKN/(B/K/N_{11})]$ are EP, further $N(B/K/N_{11})\subseteq N\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}$ and $N[BKN/(B/K/N_{11})]^{*}\subseteq\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}^{*}$,

$\begin{bmatrix}B/K/N_{11} & [(B/K/N_{12})\ (B/K/N_{13})]\\ 0 & [BKN/(B/K/N_{11})]\end{bmatrix}$ is EP iff $(B/K/N_{11})$ and $[BKN/(B/K/N_{11})]$ are EP, further $N(B/K/N_{11})\subseteq N\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}^{*}$ and

$N[BKN/(B/K/N_{11})]\subseteq N[(B/K/N_{12})\ (B/K/N_{13})]$. This proves the equivalence of (ii) and (iii). The proof is complete.

Theorem3.2

Let BKN be a matrix of the form (3) with $N(B/K/N_{11})^{*}\subseteq N[(B/K/N_{12})\ (B/K/N_{13})]^{*}$ and $N[BKN/(B/K/N_{11})]^{*}\subseteq N\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}^{*}$, then the following are equivalent.

(i) BKN is an EP matrix.

(ii) $(B/K/N_{11})$ and $[BKN/(B/K/N_{11})]$ are EP, further $N(B/K/N_{11})\subseteq N\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}$ and $N[BKN/(B/K/N_{11})]\subseteq N[(B/K/N_{12})\ (B/K/N_{13})]$;

(iii) Both the matrices $\begin{bmatrix} B/K/N_{11} & 0 \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} & \begin{bmatrix} BKN / & B/K/N_{11} \end{bmatrix} \end{bmatrix}$ and $\begin{bmatrix} B/K/N_{11} & B/K/N_{12} \quad B/K/N_{13} \\ 0 & \begin{bmatrix} BKN / & B/K/N_{11} \end{bmatrix} \end{bmatrix}$

are EP.

Proof

Theorem 3.2 follows immediately from theorem 3.1 and from the fact that BKN is EP iff BKN^* is EP. In the special case when $B/K/N_{12} \quad B/K/N_{13} = \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}^*$ we get the following.

Corollary3.3

Let $BKN = \begin{bmatrix} B/K/N_{11} & \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^* \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} & \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix} \end{bmatrix}$ with $N \; B/K/N_{11} \subseteq N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$ and

$N\begin{bmatrix} BKN / & B/K/N_{11} \end{bmatrix} \subseteq N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$, then the following are equivalent.

(i) BKN is on EP matrix;

(ii) $B/K/N_{11}$ and $\begin{bmatrix} BKN / & B/K/N_{11} \end{bmatrix}$ are EP matrix.

(iii) The matrix $\begin{bmatrix} B/K/N_{11} & 0 \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} & \begin{bmatrix} BKN / & B/K/N_{11} \end{bmatrix} \end{bmatrix}$ is EP.

Remark3.4

The condition taken on BKN is the previous Theorems are essential. This is illustrated in the following example. Let

$$K = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} \quad K = \begin{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} & \begin{bmatrix} 3 & -3 \\ 4 & 3 \end{bmatrix} \\ \begin{bmatrix} 4 & -4 \\ 5 & 4 \end{bmatrix} & \begin{bmatrix} 3 & -3 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \\ \begin{bmatrix} 3 & -3 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} & \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} \end{bmatrix}$$

$$B = \begin{bmatrix} K/N_{11} & K/N_{12} & K/N_{13} \\ K/N_{21} & K/N_{22} & K/N_{23} \\ K/N_{31} & K/N_{32} & K/N_{33} \end{bmatrix}$$

$$B=\begin{bmatrix}\begin{bmatrix}40 & 198\\ 130/3 & 238/3\end{bmatrix} & \begin{bmatrix}12 & 119/5\\ 13 & 2/5\end{bmatrix} & \begin{bmatrix}198/7 & 34/3\\ 34/3 & 4/21\end{bmatrix}\\ \begin{bmatrix}10/3 & 33/2\\ 5/72 & 10/3\end{bmatrix} & \begin{bmatrix}47/7 & 238/21\\ 110/21 & 130/21\end{bmatrix} & \begin{bmatrix}198 & 238/3\\ 40 & 130/3\end{bmatrix}\\ \begin{bmatrix}130/21 & 34/3\\ 110/21 & 40/7\end{bmatrix} & \begin{bmatrix}130/3 & 4/3\\ 110/3 & 130/3\end{bmatrix} & \begin{bmatrix}119/5 & 2/5\\ 12 & 13\end{bmatrix}\end{bmatrix}$$

$$BKN=\begin{bmatrix}B/K/N_{11} & B/K/N_{12} & B/K/N_{13}\\ B/K/N_{21} & B/K/N_{22} & B/K/N_{23}\\ B/K/N_{31} & B/K/N_{32} & B/K/N_{33}\end{bmatrix}$$

$$BKN=\begin{bmatrix}\begin{bmatrix}-22.4407 & 5323.55\\ 1780.845 & 291.3725\end{bmatrix} & \begin{bmatrix}-398.506 & 5171.344\\ 31613.405 & 94.3567\end{bmatrix} & \begin{bmatrix}234219.658 & 12820.146\\ 17891.995 & 234.7063\end{bmatrix}\\ \begin{bmatrix}12167.749 & -2081940.403\\ 3159.73 & -12169.674\end{bmatrix} & \begin{bmatrix}-5057.2896 & 65637.316\\ -1312.1612 & -401174.3662\end{bmatrix} & \begin{bmatrix}-5334.595 & -291.5211\\ -20.9322 & 1780.8861\end{bmatrix}\\ \begin{bmatrix}185833.7498 & 28159.806\\ -521.6277 & 4717.3819\end{bmatrix} & \begin{bmatrix}-5261.094 & -16.7991\\ 17.2374 & 5263.716\end{bmatrix} & \begin{bmatrix}-817.372 & -94.3564\\ -398.4684 & 31612.392\end{bmatrix}\end{bmatrix}$$

The Rank of *BKN* is 6.Hence *BKN* is EP_6.

$$\left[BKN/\ B/K/N_{11}\right]=\begin{bmatrix}-.0245 & .2091 & 9.1710 & .5022\\ -.0058 & .0028 & .0528 & .0032\\ -.3303 & -.0007 & -.1758 & -.0019\\ .009 & .0001 & .0207 & .0020\end{bmatrix}.$$

Clearly $B/K/N_{11}$ and $\left[BKN/\ B/K/N_{11}\right]$ are EP. $N\ B/K/N_{11}\subseteq N\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}$ and $N\ B/K/N_{11}{}^{*}\subseteq N\left[B/K/N_{12}\quad B/K/N_{13}\right]^{*}$. But $N\ BKN/B/K/N_{11}\not\subseteq N\left[B/K/N_{12}\quad B/K/N_{13}\right]$ and $N\ BKN/B/K/N_{11}{}^{*}\not\subseteq N\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}^{*}$, further $\begin{bmatrix}B/K/N_{11} & 0\\ \begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix} & \left[BKN/\ B/K/N_{11}\right]\end{bmatrix}$ and $\begin{bmatrix}B/K/N_{11} & \left[B/K/N_{12}\quad B/K/N_{13}\right]\\ 0 & \left[BKN/\ B/K/N_{11}\right]\end{bmatrix}$ are not EP. Thus theorem 3.1 and 3.2 as well as corollary 3.3 fail.

Theorem 3.5

Let *BKN* be of the form (3) with $\rho\ BKN=\rho\ B/K/N_{11}=r.$ Then *BKN* is an EP_r matrix if and only if *BKN* is EP_r and $\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix}B/K/N_{11}{}^{\dagger}=\left[B/K/N_{11}{}^{\dagger}\left[B/K/N_{12}\quad B/K/N_{13}\right]\right]^{*}.$

Proof

Since $\rho\ BKN=\rho\ B/K/N_{11}=r,$ we have by reason of the corollary of theorem 1 in [5], that $N\ B/K/N_{11}\subseteq N\begin{bmatrix}B/K/N_{21}\\ B/K/N_{31}\end{bmatrix},$ $N\ B/K/N_{11}{}^{*}\subseteq N\left[B/K/N_{12}\quad B/K/N_{13}\right]^{*}$ and

$$\left[BKN/\ B/K/N_{11}\right]=\begin{bmatrix}B/K/N_{22} & B/K/N_{23}\\ B/K/N_{32} & B/K/N_{33}\end{bmatrix}-\begin{bmatrix}B/K/N_{21}\\ B/K/N_{32}\end{bmatrix}B/K/N_{11}{}^{\dagger}$$

$\begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix} = 0$. According to lemma 1.1 these relations are equivalent to $\begin{bmatrix} B/K/N_{21} \\ B/K/N_{32} \end{bmatrix} = \begin{bmatrix} B/K/N_{21} \\ B/K/N_{32} \end{bmatrix} (B/K/N_{11})^{\dagger} (B/K/N_{11})$,

$(B/K/N_{12} \quad B/K/N_{13}) = (B/K/N_{11})(B/K/N_{11})^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}$ and

$$\begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix} = \begin{bmatrix} B/K/N_{21} \\ B/K/N_{32} \end{bmatrix} (B/K/N_{11})^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}.$$

Let us consider the matrices

$$P = \begin{bmatrix} I & 0 \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{32} \end{bmatrix} & I \end{bmatrix}, Q = \begin{bmatrix} I & (B/K/N_{11})^{\dagger} (B/K/N_{12} \quad B/K/N_{13}) \\ 0 & I \end{bmatrix} \text{ and } L = \begin{bmatrix} B/K/N_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

P and Q are nonsingular and by assumption $\begin{bmatrix} B/K/N_{21} \\ B/K/N_{32} \end{bmatrix} (B/K/N_{11})^{\dagger} = (B/K/N_{11})^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}$ it holds $P = Q^*$.

Therefore BKN can be factorized as $BKN = PLP^*$. Since BKN is EP$_r$, consequently, L is a well EP$_r$. Hence $N(L) = N(L)^*$ and so we have according to lemma 3 of paper [11] that $N(BKN) = N\left[PLP^*\right] = N\left[PL^*P^*\right] = N(BKN)^*$. This shows that BKN is EP.

Conversely, let us assume that BKN is EP$_r$ since $BKN=PLQ$ one choice of $(BKN)^{\dagger}$ is $(BKN)^{\dagger} = Q^{\dagger} \begin{bmatrix} B/K/N_{11} & 0 \\ 0 & 0 \end{bmatrix} P^{\dagger}$ we know that $N(BKN) = N(BKN)^*$, therefore by lemma 1.1 $(BKN)^* = (BKN)^* (BKN)^{\dagger} (BKN)$ holds, e.g.,

$$(BKN)^* = \begin{bmatrix} (B/K/N_{11})^* & \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^* \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{23} \end{bmatrix}^* & \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix}^* \end{bmatrix}$$

$$= \begin{bmatrix} (B/K/N_{11})^* & \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^* \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{23} \end{bmatrix}^* & \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix}^* \end{bmatrix} \begin{bmatrix} (B/K/N_{11})^{\dagger} (B/K/N_{11}) & (B/K/N_{11})^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

or equivalently $(B/K/N_{11})^* = (B/K/N_{11})^* (B/K/N_{11})^{\dagger} (B/K/N_{11})$ and

$$\begin{bmatrix} B/K/N_{21} \\ B/K/N_{23} \end{bmatrix}^* = \begin{bmatrix} B/K/N_{21} \\ B/K/N_{23} \end{bmatrix}^* (B/K/N_{11})^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}.$$

Form $(B/K/N_{11})^* = (B/K/N_{11})^* (B/K/N_{11})^{\dagger} (B/K/N_{11})$ it follow

$N(B/K/N_{11})^* = N(B/K/N_{11})$ i.e $(B/K/N_{11})$ is EP$_r$ and therefore

$(B/K/N_{11})(B/K/N_{11})^{\dagger} = (B/K/N_{11})^{\dagger} (B/K/N_{11})$. Taking into account

$$\begin{bmatrix} B/K/N_{21} \\ B/K/N_{23} \end{bmatrix}^* = \begin{bmatrix} B/K/N_{21} \\ B/K/N_{23} \end{bmatrix}^* B/K/N_{11}{}^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}.$$

We have $\begin{bmatrix} B/K/N_{21} \\ B/K/N_{23} \end{bmatrix} B/K/N_{11}{}^{\dagger} = \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^* \begin{bmatrix} B/K/N_{11}{}^{\dagger} \end{bmatrix}^*$

$$\begin{bmatrix} B/K/N_{11}{}^{\dagger} \; B/K/N_{11} \end{bmatrix}^* = \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^* \begin{bmatrix} B/K/N_{11}{}^{\dagger} \; B/K/N_{11} \end{bmatrix}$$

$$\begin{bmatrix} B/K/N_{11}{}^{\dagger} \end{bmatrix}^* = \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}^* \begin{bmatrix} B/K/N_{11}{}^{\dagger} \end{bmatrix}^* = \begin{bmatrix} B/K/N_{11}{}^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix} \end{bmatrix}^*$$

the theorem is proved.

In the special case when *BKN* is nonsingular *BKN* is automatically EP_r and theorem 3.5 reduces to the following.

Corollary3.6. (see theorem 9 in [8])

Let *BKN* of the form (3) with *BKN* nonsingular and $\rho\ BKN = \rho\ B/K/N_{11}$. Then *BKN* is EP if and only if

$$\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger} = \begin{bmatrix} B/K/N_{11}{}^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix} \end{bmatrix}^*.$$

Corollary3.7

Let *BKN* be nxn matrix of rank *r*. Then *BKN* is EP_r if and only if every principal sub matrix of rank *r* is EP_r.

Proof

Suppose *BKN* is an EP_r matrix. Let *BKN* be any principal submatrix of *BKN* such that $\rho\ BKN = \rho\ B/K/N_{11} = r$. Then there exists permutation matrix *P* such that $\widehat{BKN} = PBKNP^{T} = \begin{bmatrix} B/K/N_{11} & \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix} \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} & \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix} \end{bmatrix}$ and $\rho\ BKN = r$. According to Lemma 3 in [11] $p.92$ $_s\ \widehat{BKN}$ is EP, now we conclude from theorem 3.5 that *BKN* is EP_r as well. Since *BKN* was arbitrary, it follows that every principal sub matrix of rank *r* is EP_r. The converse is obvious.

4. APPLICATION

We give condition under which a partitioned matrix is decomposed into complementary summands of EP matrices BKN_1 and BKN_2 are called complementary summands of *BKN* if $BKN = BKN_1 + BKN_2$ and $\rho\ BKN = \rho\ BKN_1 + \rho\ BKN_2$.

Theorem4.1

Let *BKN* of the form (3) with $\rho\ BKN = \rho\ B/K/N_{11} + \rho\begin{bmatrix} BKN/\ B/K/N_{11} \end{bmatrix}$, where

$$\begin{bmatrix} BKN/\ B/K/N_{11} \end{bmatrix} = \begin{bmatrix} B/K/N_{22} & B/K/N_{23} \\ B/K/N_{32} & B/K/N_{33} \end{bmatrix} - \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger} \begin{bmatrix} B/K/N_{12} & B/K/N_{13} \end{bmatrix}.$$

If $B/K/N_{11}$ and $\begin{bmatrix} BKN/\ B/K/N_{11} \end{bmatrix}$ are EP matrices such that $\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}$

$B/K/N_{11}{}^{\dagger} = \left[\ B/K/N_{11}{}^{\dagger}\left[BKN/\ \ B/K/N_{11}\ \right]\right]^{*}$ and $\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right]$ $\left[BKN/\ \ B/K/N_{11}\ \right]^{\dagger} = \left[\left[BKN/\ \ B/K/N_{11}\ \right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}\right]$ then BKN can be decomposed into complementary summands of EP matrices.

Proof

Let us consider the matrices

$$BKN_1 = \begin{bmatrix} B/K/N_{11} & B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right] \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}\left[\ B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}\ \right] & \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right] \end{bmatrix}$$

and

$$BKN_2 = \begin{bmatrix} 0 & \left[I -\ B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\right] \\ & B/K/N_{12}\quad B/K/N_{13} \\ \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}\left[I -\ B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}\ \right] & \left[BKN/\ \ B/K/N_{11}\ \right] \end{bmatrix}$$

.Taking into account that

$$N\ \ B/K/N_{11}\ \subseteq N\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}\ ,$$

$$N\ \ B/K/N_{11}{}^{*} \subseteq N\left[\ B/K/N_{11}\quad B/K/N_{11}{}^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix}\right]^{*}$$ and

$$\begin{aligned} \left[BKN/\ \ B/K/N_{11}\ \right] &= \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right] \\ &\quad -\left[\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}\ \right] B/K/N_{11} \\ &\quad -\left[\ B/K/N_{11}\quad B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right]\right] \\ &= \begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right] \\ &\quad -\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right] = 0 \end{aligned}$$

We obtain by the corollary after theorem 1 in [6], that $\rho\ \ BKN_1\ = \rho\ \ BKN$. Since BKN is EP, and $\left[\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}\ \right] B/K/N_{11}{}^{\dagger}$

$$= \left[\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31} \end{bmatrix} B/K/N_{11}{}^{\dagger}\ \right] = \left[\ B/K/N_{11}{}^{\dagger}\left[\ B/K/N_{12}\quad B/K/N_{13}\ \right]\right]^{*}$$

$= \left[\ B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}\quad B/K/N_{11}{}^{\dagger}\ \ B/K/N_{12}\quad B/K/N_{13}\ \right]^{*}$, we have from theorem 3.5 that BKN_1 is EP. Since $\rho\ \ BKN_1\ = \rho\ \ B/K/N_{11}\ + \rho\left[BKN/\ \ B/K/N_{11}\ \right]$, theorem 1 of paper [6] gives

$$N\left[BKN/\ B/K/N_{11}\right]\subseteq N\left[\left[I-\ B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\right]\left[\ B/K/N_{12}\ \ B/K/N_{13}\ \right]\right]$$

$$N\left[BKN/\ B/K/N_{11}\right]\subseteq N\left[I-\ B/K/N_{11}{}^{\dagger}\right]\begin{bmatrix} B/K/N_{21} \\ B/K/N_{32}\end{bmatrix}^{*} \text{ and}$$

$$\left[I-\ B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\right]\left[BKN/\ B/K/N_{11}\right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{32}\end{bmatrix}\left[I-\left[\ B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}\ \right]\right]=0.$$

Thus by the corollary of the just applied theorem 1 in [6], we have $\rho\ BKN_2\ =\rho\left[BKN/\ B/K/N_{11}\right]$.

Further, using $B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}=\ B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}$, we obtain

$$\left[I-\ B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\right]\left[\ B/K/N_{12}\ \ B/K/N_{13}\ \right]\left[BKN/\ B/K/N_{11}\right]^{\dagger}$$

$$=\left[I-\ B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\right]\left[\left[BKN/\ B/K/N_{11}\right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{31}\end{bmatrix}\right]^{*}$$

$$=\left[\left[BKN/\ B/K/N_{11}\right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{32}\end{bmatrix}\right]^{*}\left[I-\ B/K/N_{11}\ \ B/K/N_{11}{}^{\dagger}\right]$$

$$=\left[\left[BKN/\ B/K/N_{11}\right]^{\dagger}\begin{bmatrix} B/K/N_{21} \\ B/K/N_{32}\end{bmatrix}\left[I-\ B/K/N_{11}{}^{\dagger}\ \ B/K/N_{11}\ \right]\right]^{*}.$$

Thus by theorem 3.5 BKN_2 is also EP. Clearly $BKN=BKN_1+BKN_2$, where both BKN_1 and BKN_2 are EP matrix and $\rho\ BKN\ =\rho\ B/K/N_{11}\ +\rho\left[BKN/\ B/K/N_{11}\right]$ $=\rho\ BKN_1\ +\rho\ BKN_2$. Hence BKN_1 and BKN_2 are complementary summands of EP matrices.

Remark 4.2

Any matrix that is represented as the sum of complementary summands of EP matrices is itself EP. For if $BKN=\sum_{i=1}^{k}BKN_i$ such that each BKN_i is EP and $\rho\ BKN\ =\sum rk\ BKN\ _i$ then

$$N\ BKN\ =\bigcap_{i=1}^{k}N\ BKN\ _i=\bigcap_{i=1}^{k}N\ BKN\ _i^{*}=N\ BKN\ ^{*}.$$

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