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A Fixed Point Theorem in Generalized Metric Spaces

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Abstract: *The aim of this paper is to prove a fixed point theorem in generalized metric space which generalizes the result of Akbar Azam and Muhmmad Arshad [2] Theorem [2.1]. The result is supported through an example.*

Keywords: *Complete generalized metric space, fixed point.*

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1. INTRODUCTION

In literature, the basic fixed point theorem is Banach contraction principle which asserts that if *M* is a complete metric space and *T*: $M \to M$ is a contraction mapping. That is there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq k\, d(x, y) \text{for all } x, y \in M$$

Then *T* has unique fixed point.In 2000 A. Brainciari [1] introduced generalized metric space as follows.

Definition 1.1: Let *M*be a non-empty set. Suppose that $d{:}M \times M \to R$satisfies

(1) $d(x, y) \geq 0$ for all $x, y \in M$and $d(x, y) = 0 \iff x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in M$;

(3) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$for all*x*, $y \in M$ and for all distinct points $w, z \in M - \{x, y\}$. [Rectangular property].

Then *d* is called a generalized metric and (*M*, *d*) is called a generalized metricspace.It is note that every metric space is a generalized metric space. But converseis not true. It is also illustrated with an example by A. Azam and M. Arshad with an example [1.2] in [2].

Definition 1.2: Let (M, d) be a generalized metric space. Let $\{x_n\}$be a sequence in M and $x \in M$. If for $\epsilon > 0$ there is an $n_0 \in N$such that $d(x_n, x) < \epsilon$for all $n > n_0$, then $\{x_n\}$is said to be convergent, $\{x_n\}$converges to x and *x*is the limit of $\{x_n\}$. We denote this by$lim_{n \to \infty} x_n = x$ or $\{x_n\} \to x$as $n \to \infty$also illustrated with an example by A. Azam and M. Arshad with an example [1.2] in [2].

Definition 1.3: A sequence $\{x_n\}$is said to be Cauchy sequence if for every $\epsilon > 0$ there is an$n_0 \in N$such that $d(x_n, x_{n+m}) < \epsilon$for all $n > n_0$, then $\{x_n\}$is called a Cauchy sequence in *M*.

Definition1.4: Let (M, d) be a generalized metric space. If every Cauchysequence in *M*is convergent in *M*, then *M* is called a complete generalized metricspace.
Remark that $d(a_n, y) \to d(a, y)$ and $d(x, a_n) \to d(x, a)$ when ever $\{a_n\}$is asequence in *M* with $\{a_n\} \to a$.

In [2] Akbar Azmar proved the following result on generalized metric space.

Theorem 1.5: Let (M, d) be a complete generalized metric space and themapping T: M → M satisfies the following inequality

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty) \text{where} \lambda \in [0, 1)$$

Then T has a unique fixed point.

2. MAIN RESULT

We have obtained the following main result.

Theorem 2.1: Let (M, d)be a complete generalized metric space and themapping T:M → M satisfies the following inequality

$$d(Tx, Ty) \leq \ k\ max\{d(x, y), d(x, Tx), d(y, Ty)\} \qquad \cdots\ (2.1.1)$$

for all $x, y\ \in\ M$ and $k\ \in\ [0,1)$then T has a unique fixed point in M.

Proof: Let x_0 be an arbitrary point in M. Let $x_1\ =\ T(x_0)$.

If $x_1 = x_0$ then $T(x_0)\ =\ x_0$ implies $x_0$is a fixed point of T, which shows we have nothing to prove.We now assume that $x_1 \neq\ x_0$. Let $x_2 = T(x_1)$.Define $\{x_n\}$of points in M as follows;

$$x_{n+1} = T(x_n) = T^{n+1}(x_0) \forall\ n\ =\ 1, 2, 3, \cdots$$

Using the inequality (2.1.1) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= \ d(Tx_{n-1}, Tx_n) \\ &\leq\ k \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}) \\ &\leq\ k\ [d(x_{n-1}, x_n) +\ d(x_n, x_{n+1})] \end{aligned}$$

Hence we have

$$(1\ -\ k) d(x_n, x_{n+1}) =\ k\ d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq \frac{k}{1-k} d(x_{n-1}, x_n)$$

Suppose that x_0 is not a periodic point. Infact if $x_n = x_0$then

$$\begin{aligned} d(x_0, Tx_0)\ &=\ d(x_n,\ Tx_n)\ =\ d(T^{\ n}x_0,\ T^{\ n+1}x_0) \\ &\leq \left(\frac{k}{1-k}\right) d(T^{n-1}x_0,\ T^n x_0) \\ &\leq \left(\frac{k}{1-k}\right)^2 d(T^{n-2}x_0,\ T^{n-1}x_0) \\ &\vdots \\ &\leq \left(\frac{k}{1-k}\right)^n d(x_0, T\, x_0) \end{aligned}$$

put$h\ =\ \frac{k}{1-\ k}$then $h\ <\ 1$ and $(1 - h^n)\ d(x_0, T\, x_0) \leq 0$.

Which shows $x\ _0$is a fixed point of T.

Thus we can suppose that$x_n \neq\ x_0$. *i.e.* $T^n x_0 \neq x_0 \forall\ n\ =\ 1, 2, 3,\ \cdots$

Now from the inequality

$$d(T^n x_0, T^{n+m}\ x_0) \leq\ k \max\{d(T^{n-1}x_0,\ T^{\ n+m-1}x_0), d(T^{\ n-1}x_0,\ T^n x_0),$$

$$d(T^{n+m-1}\ x_0,\ T^{\ n+m}x_0)\}$$

$$\leq k \max\{h^{n-1} d(x_0, T^m x_0), h^{n-1} d(x_0, Tx_0), h^{n+m-1} d(x_0, Tx_0)\}$$

Therefore $d(x_n, x_{n+m}) \to 0$ as $n \to \infty$ *i.e.* $\{x_n\}$is a Cauchy sequence. Since$M$ is a complete generalized metric space, there exists a $u \in M$ such that$x_n \to u$.
By rectangular property we have

$$d(Tu, u) \leq d(Tu, T^n x_0) + d(T^n x_0, T^{n+1} x_0) + d(T^{n+1} x_0, u)$$
$$\leq k\, max\{d(u, T^{n-1} x_0), d(u, Tu), d(T^{n-1} x_0, T^n x_0)\} + h^n d(x_0, Tx_0) + d(T^{n+1} x_0, u)$$

As $n \to \infty$and using remark we have $u = Tu$.
Now we have to show that T has a unique fixed point. Suppose there existsone another fixed point v in M such that $v = Tv$.

Now

$$d(u, v) = d(Tu, Tv)$$
$$\leq k\, max\{d(u, v), d(u, Tu), d(v, Tv)\}$$
$$\leq k\, d(u, v)$$
$$\Rightarrow (1 - k)\, d(u, v) \leq 0$$
$$\Rightarrow d(u, v) = 0$$

Hence $u = v$.

Now we give an example for our result which satisfies our inequality and 1as a unique fixed point of T.

Example 2.2: Let $M = \{0, 1, 2, 3\}$ and define $d : M \times M \to \mathbb{R}$by

$$d(0,2) = d(0,3) = d(3,0) = d(2,0) = d(2,3) = d(3,2) = 3.$$
$$d(1,2) = d(1,3) = d(2,1) = d(3,1) = 1$$
$$d(0,1) = d(1,0) = 1$$

Also define $d(x, y) = 0 \Leftrightarrow x = y\ \forall x, y \in M$

It is observed that (M, d) is a generalized metric space, but not a metric space.Because it lacks triangular inequality.

$$d(2,0) = 3.\text{But } d(2,1) + d(1,0) = 2.\ i.e.\ d(2,0) > d(2,1) + d(1,0).$$

Define $T : M \to M$ by $T(x) = \begin{cases} 1, & if\ x \neq 3 \\ 0, & if\ x = 3 \end{cases}$

Then clearly Tsatisfies our inequality and has unique fixed point 1.

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