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A Special case of Complemented Almost Distributive Lattices

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Abstract: *Derive equivalent conditions for an Almost Distributive Lattice(ADL) to become a complemented ADL in terms of m-filters.*

Keywords: *Almost Distributive Lattice (ADL), complemented ADL, complemented element, m-filter, prime ideal, maxima filter*.

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1. Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [6] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set PI(L) of all principal ideals of L forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. Swamy, G.C. Rao and G.N. Rao introduced the concept of Stone ADL and characterized it in terms of its ideals. In [5], SambasivaRao and G.C. Rao introduced O-ideal in Almost Distributive lattices and proved their properties. In this paper we characterize the complemented ADL with help of m-filters.

2. Preliminaries

Definition 2.1.[6] An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type (2, 2, 0) satisfying

1. $(x\vee y) \wedge z = (x \wedge z) \vee(y \wedge z)$
2. $x\wedge(y \vee z) = (x \wedge y) \vee(x \wedge z)$
3. $(x\vee y) \wedge y = y$
4. $(x\vee y) \wedge x = x$
5. $x\vee(x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x\vee 0 = x$, for any $x, y, z \in L$.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0\in X$. Define the binary operations $\vee$, $\wedge$on X by

$$x \vee y = \begin{cases} x & \text{if} \quad x \neq x_0 \\ y & \text{if} \quad x = x_0 \end{cases} \quad x \wedge y = \begin{cases} y & \text{if} \quad x \neq x_0 \\ x_0 & \text{if} \quad x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL. If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.2: ([6]) *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in L*

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$

(10) $a \wedge a = a$ *and* $a \vee a = a$

(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$

(12) If $a \leq c$, $b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

(13) $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice.

Theorem 2.3. ([6]) *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*

(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice*

(2) $a \vee b = b \vee a$, *for all* $a, b \in L$

(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$

(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.4: ([6]) *Let L be an ADL and $m \in L$. Then the following are equivalent:*

(1) m is maximal with respect to $\leq$

(2) $m \vee a = m$, *for all* $a \in L$

(3) $m \wedge a = a$, *for all* $a \in L$

(4) $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [[1], [2]], a non-empty sub set I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$. The set I(L) of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by

$(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write (s] instead of (S]. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write [s) instead of [S).

Theorem 2.5 ([6]). *For any* x, y *in* L *the following are equivalent:*

1). $(x] \subseteq (y]$

2). $y \wedge x = x$

3). $y \vee x = y$

4). $[y) \subseteq [x)$.

For any x, y $\in$ L, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set PI(L) of all principal ideals of L is a sublattice of the distributive lattice I(L) of ideals of L.

For any $A \subseteq L$, $A^* = \{ x \in L \mid a \wedge x = 0 \text{ for all } a \in A \}$ is an ideal of L. We write $(a]^*$ for $\{a\}^*$ and this is called an annulet of L. Clearly $(0]^* = L$ and $L^* = (0]$. For any $X \subseteq L$, Let us define $X^+ = \{y \in L \mid x \vee y \text{ is maximal element for all } x \in X\}$. We write $(x)^+$ for $\{x\}^+$, for any $x \in L$. Clearly $(x)^+ = L$ if and only if x is maximal and also we have $(0)^+$ is the set of all maximal elements of L.

3. CHARACTERIZATION OF COMPLEMENTED ADLS

In this paper we characterize the complemented ADL in terms of $m-$filters.

We begin with the following definition

Definition 3.1: Let L be an ADL with maximal elements. An element x of an ADL L is said to be complemented if there exists an element y $\in$L such that $x \wedge y = 0$ and $x \vee y$ maximal. An ADL L is said to be a complemented ADL if every element of L is a complemented element.

Now we prove the following

Theorem 3.2: *Let L be an ADL with maximal elements. Then L is complemented ADL if and only if the set of all prime filters of L is unordered.*

Proof: Assume that L is complemented ADL. Then every prime filter of L is a maximal filter and hence the set of all prime filters of L is unordered. Conversely, assume that the set of all prime filters of L is unordered. We prove that L is complemented ADL. Suppose that L is not a complemeted ADL. Then there exists a$\in$ L which is non-zero and non maximal element such that $(a)^* \cap (a)^+ = \phi.$ Consider $I = (a)^* \vee (a].$ Then clearly I is an ideal of L. If a maximal element $m \in I$ then $m = a \vee b,$ for some $b \in (a)^*$. That implies $b \in (a)^+$ and hence $b \in (a)^* \cap (a)^+.$Therefore $(a)^* \cap (a)^+ \neq \phi,$ which is a contradiction. Thus $m \notin I$ and hence I is a proper ideal of L. Then there exists a maximal ideal G of L such that $I \subseteq G$. Since G is a maximal ideal, we get that G is a prime ideal of L. That implies that L\G is a prime filter of L. Since $I = (a)^* \vee (a]$, we get $a \in I$ and hence $a \notin L \setminus G$. Consider $(L \setminus G) \vee [a) = Q.$ Clearly Q is a proper filter of L. So that Q must be contained in some maximal filter M of L. Since $a \in Q,$ we get that $a \in M$ and hence $L \setminus G \underset{\neq}{\subset} M.$ This is not possible because the set of all prime filters of L is unordered. Hence $(a)^* \cap (a)^+ \neq \phi.$ Thus L is complemented ADL.

The following theorem can be verified easily.

Theorem 3.3: *Let L be an ADL with maximal elements. Then L is complemented if and only if every maximal filter of L is a minimal prime filter.*

Now, we have the following

Theorem 3.4: *Let L be an ADL with maximal elements. Then L is complemented ADL if and only if for any maximal filter M of L, we have* $M = \{x \in L \mid x \vee y \text{ is maximal, for some } y \notin M\}.$

Proof: Assume that L is a complemented ADL. Let M be any maximal filter of L. Let $x \in M$. Since L is a complemented ADL, there exists an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is maximal. Since M maximal filter, we get $y \notin M$. Therefore $M \subseteq \{x \in L \mid x \vee y$ is maximal, for some

$y \notin M\} = Q.$ Let $x \in Q.$ Then $x \vee y$ is maximal, for some $y \notin M.$ Since M is a prime filter of L and $y \notin M$, we get that $x \in M.$ Hence $Q = M.$ Conversely assume that for any maximal filter M of an ADL L we have $M = \{x \in L \mid x \vee y$ is maximal, for some $y \notin M\}.$ We prove that L is a complemented ADL. Suppose that L is not complemented. Then there exists an element $a \in L$ which is non zero and non maximal element. Consider $F = [a) \vee (a)^+$. If F is a proper filter of L. Then there exists a maximal filter M of L such that $F \subseteq M$. That implies $a \in M.$ By our assumption, there exists an element $b \notin M$ such that $a \vee b$ is maximal. That implies $b \in (a)^+.$ Therefore $b \in M$, which is a contradiction. Hence F is not a proper filter of L. Thus F=L. Then $0 \in F.$ That implies $0 = a \wedge b,$ for some $b \in (a)^+.$ Therefore $0 = a \wedge b$ and $a \wedge b$ is maximal. Hence a is a complemented element of L. Thus L is a complemented ADL.

Definition 3.5: Let L be an ADL with maximal elements. For any ideal I of an ADL L, define $m(\mathrm{I}) = \{\mathrm{x} \in \mathrm{L} \mid \mathrm{x} \vee \mathrm{y}$ is maximal, for some $y \in I\}.$

We prove the following result.

Lemma 3.6: Let L be an ADL with maximal elements. For any two ideals I, J of an ADL $L,$ we have the following:

1. $m(\mathrm{I})$ is a filter of L
2. $m(\mathrm{I}) = \bigcup_{x \in I} (\mathrm{x})^+$
3. $I \subseteq J \Rightarrow m(\mathrm{I}) \subseteq m(\mathrm{J})$
4. $m(\mathrm{I} \cap J) = m(\mathrm{I}) \cap m(\mathrm{J})$
5. $m(\mathrm{I}) \vee m(\mathrm{J}) \subseteq m(\mathrm{I} \vee J).$

Proof: 1. Clearly, we have $m(\mathrm{I}) \neq \phi.$ Let $x, y \in m(\mathrm{I}).$ Then $x \vee i$ and $y \vee j$ are maximal elements, for some $i, j \in I.$ For any $s \in L,$ $[(\mathrm{x} \wedge \mathrm{y}) \vee (\mathrm{i} \vee \mathrm{j})] \wedge \mathrm{s} = (\mathrm{x} \vee \mathrm{i} \vee \mathrm{j}) \wedge (\mathrm{y} \vee \mathrm{i} \vee \mathrm{j}) \wedge \mathrm{s} = s.$

Therefore $(\mathrm{x} \wedge \mathrm{y}) \vee (\mathrm{i} \vee \mathrm{j})$ is maximal and hence $\mathrm{x} \wedge \mathrm{y} \in m(\mathrm{I}).$ Let $x \in m(\mathrm{I})$ and $r \in L.$ Then $\mathrm{x} \vee \mathrm{i}$ is maximal, for some $i \in I.$ Hence $r \vee x \vee i$ is a maximal element of $L.$ Therefore $r \vee x \in m(\mathrm{I}).$ Thus $m(\mathrm{I})$ is a filter of an ADL $L.$

2. Let $a \in m(\mathrm{I}).$ Then there exists $b \in I$ such that $a \vee b$ is a maximal element of $L.$ That implies $a \in (\mathrm{b})^+.$ Therefore $m(\mathrm{I}) \subseteq \bigcup_{b \in I} (\mathrm{b})^+.$ Conversely, let $x \in \bigcup_{b \in I} (\mathrm{b})^+.$ Then there exists an element $y \in I$ such that $x \in (\mathrm{y})^+.$ That implies $\mathrm{x} \vee \mathrm{y}$ is maximal. So that $x \in m(\mathrm{I}).$ Therefore $\bigcup_{b \in I} (\mathrm{b})^+ \subseteq m(\mathrm{I}).$ Hence $m(\mathrm{I}) = \bigcup_{b \in I} (\mathrm{b})^+.$

3. Obvious.

4. Since $I \cap J \subseteq I$, $I \cap J \subseteq J$, we get that $m(\mathrm{I} \cap J) \subseteq m(\mathrm{I})$ and $m(\mathrm{I} \cap J) \subseteq m(\mathrm{J}).$ Hence $m(\mathrm{I} \cap J) \subseteq m(\mathrm{I}) \cap m(\mathrm{J}).$ Let $x \in m(\mathrm{I}) \cap m(\mathrm{J}).$ Then there exist $y \in I$ and $z \in J$ such that $x \vee y, x \vee z$ are maximal elements. Now, for any $s \in L,$ $[\mathrm{x} \vee (\mathrm{y} \wedge \mathrm{z})] \wedge \mathrm{s} = [(\mathrm{x} \vee \mathrm{y}) \wedge (\mathrm{x} \vee \mathrm{z})] \wedge \mathrm{s} = \mathrm{s}.$

Therefore $x \vee (\mathrm{y} \wedge \mathrm{z})$ is maximal. Since $y \wedge z \in I \cap J,$ we get that $x \in m(\mathrm{I} \cap J)$ and hence $m(\mathrm{I}) \cap m(\mathrm{J}) \subseteq m(\mathrm{I} \cap J).$ Therefore $m(\mathrm{I} \cap J) = m(\mathrm{I}) \cap m(\mathrm{J})$.

5. Let $x \in m(\mathrm{I}) \vee m(\mathrm{J}).$ Then there exist $a \in m(\mathrm{I}), \mathrm{b} \in m(\mathrm{J})$ such that $x = a \wedge b.$ Since $a \in m(\mathrm{I}), \mathrm{b} \in m(\mathrm{J})$, there exist $f \in I, g \in J$ such that $a \vee f, \mathrm{b} \vee \mathrm{g}$ are maximal elements. Clearly, we have $f \vee g \in I \vee J.$ Now, for any $s \in L,$

$[x\vee(f\vee g)]\wedge s=[(a\wedge b)\vee(f\vee g)]\wedge s$

$=(a\vee f\vee g)\wedge(b\vee f\vee g)\wedge s=s$

Therefore $x\vee(f\vee g)$ is maximal. Hence $m(I)\vee m(J)\subseteq m(I\vee J)$.

Theorem 3.7: Let $\{I_\alpha\}_{\alpha\in\Delta}$ be an family of ideals of an ADL L with maximal elements. Then $m(\bigcap_{\alpha\in\Delta} I_\alpha)=\bigcap_{\alpha\in\Delta} m(I_\alpha)$.

Proof: Let $x\in m(\bigcap_{\alpha\in\Delta} I_\alpha)$. Then there exists an element $y\in\bigcap_{\alpha\in\Delta} I_\alpha$ such that $x\vee y$ is maximal. That implies that $x\vee y$ is maximal and $y\in I_\alpha$, for all $\alpha\in\Delta$. Therefore $x\in m(I_\alpha)$, for all $\alpha\in\Delta$. Hence $x\in\bigcap_{\alpha\in\Delta} m(I_\alpha)$. Thus $m(\bigcap_{\alpha\in\Delta} I_\alpha)\subseteq\bigcap_{\alpha\in\Delta} m(I_\alpha)$. Conversely, assume that $x\in\bigcap_{\alpha\in\Delta} m(I_\alpha)$. Then $x\in m(I_\alpha)$, for all $\alpha\in\Delta$. Since $x\in m(I_\alpha)$, there exists an element $y\in I_\alpha$ such that $x\vee y$ is maximal. That implies $x\vee y$ is maximal and $y\in\bigcap_{\alpha\in\Delta} I_\alpha$. Therefore $x\in m(\bigcap_{\alpha\in\Delta} I_\alpha)$ and hence $\bigcap_{\alpha\in\Delta} m(I_\alpha)\subseteq m(\bigcap_{\alpha\in\Delta} I_\alpha)$. Thus $m(\bigcap_{\alpha\in\Delta} I_\alpha)=\bigcap_{\alpha\in\Delta} m(I_\alpha)$.

Lemma 3.8: Let L be an ADL with maximal elements. Then $m(I)=L$ if and only if $I=L$.

Proof: Assume that $m(I)=L$. then we can choose $0\in m(I)$. That implies there exists an element $y\in I$ such that $y=y\vee 0$ is maximal. Since y is maximal and $y\in I$, we get that $I=L$. Conversely, assume that $I=L$. Choose a maximal element n of an ADL L such that $n\in I$. Therefore $x\vee n$ is maximal, for all $x\in L$. Hence $m(I)=L$.

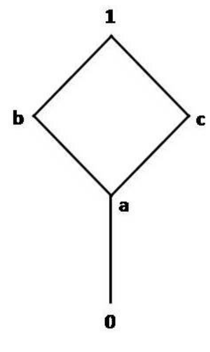
We have following definition.

Definition 3.9: A filter F of an ADL L is said to be $m-$filter if $m(I)=F$, for some ideal I of L.

Example 3.10: Let $A=\{0,a\}$ and $B=\{0,b_1,b_2\}$ be two discrete ADLs. Write $L=A\times B=\{(0,0),(0,b_1),(0,b_2),(a,0),(a,b_1),(a,b_2)\}$. Then $(L,\vee,\wedge,0^{'})$ is an ADL, where $0^{'}=(0,0)$ under the pointwise operations. Consider the ideal $I=\{(0,0),(0,b_1),(0,b_2)\}$ and filter $F=\{(a,0),(a,b_1),(a,b_2)\}$. Now, $m(I)=\bigcup_{x\in I}(x)^+=\{(a,0),(a,b_1),(a,b_2)\}=F$.

Therefore F is an $m-$filter of L.

Example 3.11: Let $L=\{0,a,b,c,1\}$ be a distributive lattice whose Hasse diagram is given in the following.



Consider $F=\{1,b\}$ and $I=\{0,a,c\}$. Clearly, F is a filter and I is an ideal of L. Now,

$m(I)=(0)^+\cup(a)^+\cup(c)^+$

$=\{1\}\cup\{1\}\cup\{b,1\}=\{b,1\}=F$.

Therefore F is an $m-$filter. But $G=\{\mathrm{a,b,c,1}\}$ is not a $m-$filter.

We characterize the complemented ADL in terms of $m-$filters.

Theorem 3.12: *Let L be an ADL with maximal elements. Then L is a complemented ADL if and only if every maximal filter of L is* $m-$*filter.*

Proof: Assume that L is a complemented ADL. Let M be any maximal filter of L. By the above theorem, we have $M=\{x\in L\,|\,x\vee y$ is maximal, for some $y\notin M\}$. Hence M is a $m-filter$ of L. Conversely assume that every maximal filter of L is a $m-filter$ of L. Suppose that L is not a complemented ADL. Then there exists an element a in L which is non zero and non maximal element. Consider $F=[a)\vee(a)^{+}$. Clearly Fis a filter of L. By the above theorem 3.4, we have L is complemented ADL.

Finally, we conclude this paper with the following result.

Therorem 3.13: *Let L be an ADL with maximal elements. Then the following are equivalent:*

1. *L is complemented ADL*
2. *Every prime ideal of L is maximal*
3. *Every prime filter of L is maximal*
4. *Complement of a maximal ideal of L is a maximal filter of L*
5. *Complement of a maximal filter of L is a maximal ideal of L*
6. *The set of all prime ideals of L is unordered*
7. *The set of all prime filters of L is unordered*
8. *Any maximal ideal M of L is expressed as* $M=\{x\in L\,|\,x\wedge y=0,$ *for some* $y\notin M\}$
9. *Any maximal filter M of L is expressed as* $M=\{x\in L\,|\,x\vee y$ *is a maximal element, for some* $y\notin M\}$
10. *Every maximal filter of L is a* $m-$*filter of L*
11. *Every maximal ideal of L is minimal prime ideal of L*
12. *Every maximal filter of L is a minimal prime filter of L.*

4. Conclusion

In this paper, some fruitful properties on $m-$filters are established which based on complemented ADL.

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