International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 11, November 2015, PP 1-8
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Equivariant Bifurcation of a Coupled Feedback Jerk System with Delay

Chunrui Zhang
Department of Mathematics
Northeast Forestry University
Harbin, China
math@nefu.edu.cn

Hongpeng Li
College of Mechanical and electrical engineering
Northeast Forestry University
Harbin, China
lihongpengsy@163.com

Ye Sun
Department of Mathematics
Northeast Forestry University
Harbin, China
616292301@qq.com

Abstract: *A delay-coupled jerk system is considered. These structures are based on coupled mutually oscillators which oscillate at the same frequency. By taking the time delay as a bifurcation parameter, the stability of the zero equilibrium and the existence of Hopf bifurcations induced by delay are investigate, and then the direction and stability of the Hopf bifurcations are studied. Using the symmetric functional differential equation theories, the multiple Hopf bifurcations of the equilibrium are demonstrated. The existence of multiple branches of bifurcating periodic solutions and their spatio-temporal patterns are obtained. Some numerical simulations are used to illustrate the effectiveness of the obtained results*.

Keywords: *coupled jerk system, symmetric bifurcation, stability, spatio-temporal, delay.*

1. INTRODUCTION

A jerk equation is an autonomous third-order differential equation of the form

$$\frac{d^3x}{dt^3} = f\left(\frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right). \tag{1.1}$$

Here, x denotes a scalar, for example the position coordinate[1,2]. It is well known that any explicit n-order scalar ordinary differential equation can be recast in the form of a system of n coupled first-order differential equations, i.e. in the form of a n-dimensional dynamical system. In particular, a scalar third-order differential equation of the form (1.1) can be transformed into the three-dimensional dynamical system:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = f(x_3(t), x_2(t), x_1(t)), \end{cases} \tag{1.2}$$

for letting

$$x_1(t) = x, x_2(t) = \frac{dx_1}{dt}, x_3(t) = \frac{dx_2}{dt}.$$

Time-delayed feedback has been introduced as a powerful tool for control of unstable periodic orbits or control of unstable steady states[3]. In Ref [3], regarding the delay as parameter, the authors investigate the effect of delay on the dynamics of Jerk system with delayed feedback:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = f(x_3(t), x_2(t), x_1(t)) + k(x_1(t) - x_1(t-\tau)), \end{cases} \tag{1.3}$$

where τ is a positive constant number.

In the paper, considering the coupling case, we attempt to analytically investigate how the coupling time delay and the coupling strength affect spatio-temporal patterns of bifurcating periodic oscillations of jerk model:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = x_1(t) - \gamma x_2(t) - x_3(t) + (x_1(t))^3 + k(x_1(t) - x_4(t-\tau)), \\ \dot{x}_4(t) = x_5(t), \\ \dot{x}_5(t) = x_6(t), \\ \dot{x}_6(t) = x_4(t) - \gamma x_5(t) - x_6(t) + (x_4(t))^3 + k(x_4(t) - x_1(t-\tau)), \end{cases} \tag{1.4}$$

where τ is a positive constant number. The results show that the jerk model can exhibit different spatio-temporal patterns sensitive to the delay. It has been shown that even small, comparing to the oscillation period, delays may have a large impact on the dynamics of delay-coupled jerk model.

The plan for the article is as follows. In section 2, we consider the linear stability of Eq.(1.4) and present some theorems about the region of stability of the trivial solution as a function of the physical parameters in the model. We find some new phenomena such as stability switch for Eq.(1.4) which is not mentioned in Ref.[3]. Couple can lead synchronization, phase trapping, phase locking, amplitude death, chaos, bifurcation of oscillators and so on[4-5]. Since two identical oscillators are coupled symmetrically, then the most typical patterns of behavior are perfect synchrony or perfect antisynchrony (in which the oscillators are half a period out of phase with each other), see Ref.[6-8]. In section 3,we give the $Z_2-equivariant$ property of Eq.(1.4) and the existence of multiple periodic solutions (synchronous respectively, anti-phased). In the final section, we present some numerical simulation to support our analysis results.

2. Stability and Bifurcation Analysis

In this section we shall analyze the distribution of the roots of the associated characteristic equation to discuss the stability and existence of Hopf bifurcation, by regarding τ as parameter.

We know that $(x^*, 0, 0, x^*, 0, 0)$ is an equilibrium of Eq.(1.3), where $x^* = \pm 1$. Let

$$\begin{cases} y_1(t) = x_1(t) - x^*, \\ y_2(t) = x_2(t), \\ y_3(t) = x_3(t), \\ y_4(t) = x_4(t) - x^*, \\ y_5(t) = x_5(t), \\ y_6(t) = x_3(t), \end{cases}$$

then Eq.(1.4) can be rewritten as

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = y_3(t), \\ \dot{y}_3(t) = y_1(t) - \gamma y_2(t) - y_3(t) - (y_1(t))^3 - 3x^* (y_1(t))^2 - 3x^{*2} y_1(t) + k(y_1(t) - y_4(t-\tau)), \\ \dot{y}_4(t) = y_5(t), \\ \dot{y}_5(t) = y_6(t), \\ \dot{y}_6(t) = y_4(t) - \gamma y_5(t) - y_6(t) - (y_4(t))^3 - 3x^* (y_4(t))^2 - 3x^{*2} y_4(t) + k(y_4(t) - y_1(t-\tau)). \end{cases} \tag{2.1}$$

The linearization of Eq.(2.1) around $(0,0,0,0,0,0)$ is

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = y_3(t), \\ \dot{y}_3(t) = (1+k-3x^{*2}) y_1(t) - \gamma y_2(t) - y_3(t) - k y_4(t-\tau), \\ \dot{y}_4(t) = y_5(t), \\ \dot{y}_5(t) = y_6(t), \\ \dot{y}_6(t) = (1+k-3x^{*2}) y_4(t) - \gamma y_5(t) - y_6(t) - k y_1(t-\tau). \end{cases} \tag{2.2}$$

The characteristic equation associated with the linearization of (2.2) is

$$\Delta = \left[\lambda^3 + \lambda^2 + \gamma\lambda + (5-k) + ke^{-\lambda\tau}\right]\left[\lambda^3 + \lambda^2 + \gamma\lambda + (5-k) - ke^{-\lambda\tau}\right] = \Delta_1 \Delta_2 = 0, \tag{2.3}$$

where

$$\Delta_1 = \lambda^3 + \lambda^2 + \gamma\lambda + (5-k) + ke^{-\lambda\tau},$$
$$\Delta_2 = \lambda^3 + \lambda^2 + \gamma\lambda + (5-k) - ke^{-\lambda\tau}.$$

The distribution of zeros of Eq.(2.3) determines the dymatic properties of (1.4). In what follows, the analysis on the distribution of the roots to Eq.(2.3) is based on the conclusion: the sum of the order of the zeros of Eq.(2.3) in an open right-plane can change only if a zero appears or accesses the imaginary axis as parameter is varied.

Consider

$$g(z) = z^3 + (1-2\gamma) z^2 + (\gamma^2 - 2(5-k)) z + 25 - 10k.$$

In the following, we make the assumption (H_1) on $g(z)$:

(H_1) If $g(z_0) = 0$, then $g'(z_0) \neq 0$.

Consider equation $\Delta_1 = 0$. Let $i\omega\ (\omega > 0)$ be a root of $\Delta_1 = 0$, then plugging $i\omega$ into $\Delta_1 = 0$ to get $-i\omega^3 - \omega^2 + \gamma i\omega + 5 - k + ke^{-i\omega\tau} = 0$. Hence,

$$\omega^6 + (1-2\gamma)\omega^4 + (\gamma^2 - 2(5-k))\omega^2 + 25 - 10k = 0 \tag{2.4}$$

Let $z^{(j)}\ (j = 1,2,3)$ be positive zeros of the equation $g(z) = 0$. Then $\omega^{(j)}\ (j = 1,2,3)$ are roots of Eq.(2.4). Define $\tau_n^{(j)}$ as (2.5) or (2.6):

$$\tau_n^{j\ (1)} = \frac{1}{\omega^j}\left(\arccos\frac{(\omega^j)^2 - 5 + k}{k} + 2n\pi\right) \quad n = 0,1,\ldots, \tag{2.5}$$

or

$$\tau_n^{j\ (1)} = \frac{1}{\omega^j}\left(\arcsin\frac{-(\omega^j)^3 + \gamma\omega^j}{k} + 2n\pi\right) \quad n = 0,1,\ldots. \tag{2.6}$$

Similarly, under the condition H_1, we have : for $\Delta_2 = 0$

$$\tau_n^{j\ (2)} = \frac{1}{\omega^j}\left(2\pi - \arccos\frac{(\omega^j)^2 - 5 + k}{k} + 2n\pi\right) \quad n = 0,1,\ldots, \tag{2.7}$$

or

$$\tau_n^{j\ (2)} = \frac{1}{\omega^j}\left(2\pi - \arcsin\frac{-(\omega^j)^3 + \gamma\omega^j}{k} + 2n\pi\right) \quad n = 0,1,\ldots. \tag{2.8}$$

Then we have the following lemma:

Lemma 2.1. If H_1 holds, then $\frac{d\alpha}{d\tau}\big|_{\tau=\tau_n^{j\ (1,2)}} \neq 0$.

Proof. Differentiating both sides of Eq.(2.3) with respect to τ and replacing τ by τ_n^j, we have

$$\operatorname{Re}\frac{d\alpha}{d\tau}\Big|_{\substack{\tau=\tau_n^{j\ (1,2)}\\ \omega=\omega^j}} = \frac{g'(z^j)}{\sqrt{(\omega^j)k^2}} \neq 0.$$

Applying the above conclusions and bifurcation theorems for functional differential equations, we have the following results presenting the stability and existence bifurcations to the symmetric system (1.4):

Theorem 2.1 The system (2.1) undergoes Hopf bifurcation when

$$\tau = \tau_n^{j\ (1,2)} \quad (j = 1,2,3; n = 1,2,\ldots).$$

3. Existence of Multiple Periodic Solutions

In the following, we consider the symmetric properities of Eq.(1.4). Using the theories of functional differential equation, (1.1) can be written as

$$\dot{x}(t) = Lx_t + Fx_t \quad n = 0,1,\ldots, \tag{3.1}$$

$$L\phi = \begin{pmatrix} A_1 & O \\ O & A_1 \end{pmatrix}\phi(0) + \begin{pmatrix} O & A_2 \\ A_2 & O \end{pmatrix}\phi(-\tau),$$

where $x_t = x(t+\theta)$ for $-\tau \le \theta \le 0$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1+k-3x^{*2} & -\gamma & -1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix},$$

and

$$F\phi = \begin{pmatrix} 0 \\ 0 \\ -6\phi_1^2 - \phi_1^3 \\ 0 \\ 0 \\ -6\phi_4^2 - \phi_4^3 \end{pmatrix}$$

for $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in C([-\tau, 0], R^6)$

It is clearly that the system (3.1) is Z_2-equivariant with

$$(\rho U)_r = U_{r+1} \pmod 2,$$

for any U_r in R^2. It is much interesting to consider the spatio-temporal patterns of bifurcating periodic solutions. For this purpose ,we give the concepts of some spatiotemporal symmetric periodic solutions. Assume that the state $(u_1(t), \upsilon_1(t), \omega_1(t), u_2(t), \upsilon_2(t), \omega_2(t))$, can possess two different types of symmetry: spatial and temporal. The oscillators $(u_1(t), \upsilon_1(t), \omega_1(t))$ and $(u_2(t), \upsilon_2(t), \omega_2(t))$ are synchronized if the state taking the form

$$(u(t), \upsilon(t), \omega(t), u(t), \upsilon(t), \omega(t)),$$

for all times t. On the other hand, oscillator $(u_1(t), \upsilon_1(t), \omega_1(t))$ is half a period out of phase with (anti-synchronous) oscillator $(u_2(t), \upsilon_2(t), \omega_2(t))$ means the state taking the form

$$\left(u(t), \upsilon(t), \omega(t), u\left(t+\frac{T}{2}\right), \upsilon\left(t+\frac{T}{2}\right), \omega\left(t+\frac{T}{2}\right)\right)$$

Now, we explore the possible (spatial) symmetry of the system (3.1). Consider the action of $Z_2 \times S^1$ on $([-\tau, 0], R^4)$ with

$$(r,\theta)x(t) = rx(t+\theta) \qquad (r,\theta) \in Z_2 \times S^1,$$

where S^1 is the temporal. Let $T = \frac{2\pi}{\omega^j}$ and denote P_T the Banach space of all continuous $T-$periodic function $x(t)$. Denoting SP_T the subspace of P_T consisting of all $T-$periodic solution of system (3.1) with $\tau = (\tau_n^j)^{1,2}$, then for each subgroup $\sum \subset Z_2 \times S^1$,

$$Fix(\sum, SP_T) = \{x \in SP, (r,\theta)x = x, \textit{ for all } (r,\theta) \in \sum\}$$

is a subspace.

Theorem 3.1 The trivial solution of system (3.1) undergoes a Hopf bifurcation at giving rise to one branch of synchronous (respectively,anti-phased) periodic solutions.

Proof. Let $\pm i\omega^j$ satisfy Eq.(2.4). The corresponding eigenvectors of $\Delta(\lambda)$ can be chosen as

$$q_1(\theta) = (\upsilon_1(\theta)^T, \upsilon_1(\theta)^T)^T,$$

where $\upsilon_1(\theta)$ satisfies $\left(A_1 + e^{-i\omega^j \tau_n^{j(1)}} A_2\right)\upsilon_1(\theta) = i\omega^j \upsilon_1(\theta)$;

$$q_2(\theta) = (\upsilon_2(\theta)^T, -\upsilon_2(\theta)^T)^T$$

and $\upsilon_2(\theta)$ satisfies $\left(A_1 + e^{-i\omega^j \tau_n^{j(2)}} A_2\right)\upsilon_2(\theta) = i\omega^j \upsilon_2(\theta)$.

The isotropic subgroup of $Z_2 \times S^1$ is $z_2(\rho)$, the center space associated to eigenvalues $\pm i\omega^j$ which implies that is spanned by $q_1(\theta)$ and $\bar{q}_1(\theta)$, the bifurcated periodic solutions are synchronous, taking the form

$$(u(t), \upsilon(t), \omega(t), u(t), \upsilon(t), \omega(t)).$$

Similarly, $Z_2 \times S^1$ has another isotropic subgroup $z_2(\rho, \pi)$, the center space associated to eigenvalues $\pm i\omega^j$ is spanned by $q_2(\theta), \bar{q}_2(\theta)$ which implies that the bifurcated periodic solutions are anti-phased, i.e. ,taking the form

$$\left(u(t), \upsilon(t), \omega(t), u\left(t+\frac{T}{2}\right), \upsilon\left(t+\frac{T}{2}\right), \omega\left(t+\frac{T}{2}\right)\right),$$

where T is a period.

4. NUMERICAL SIMULATIONS

In this section, we use some numerical simulations to illustrate the analytical results we obtained in previous sections.

Let parameters $\gamma = 2.2$.

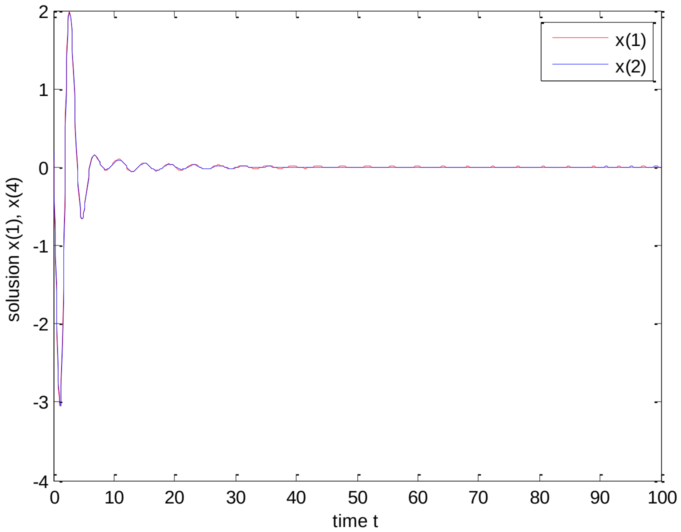

Fig1. *A branch of synchronous stable solutions appears with* $\tau = 0.1$.

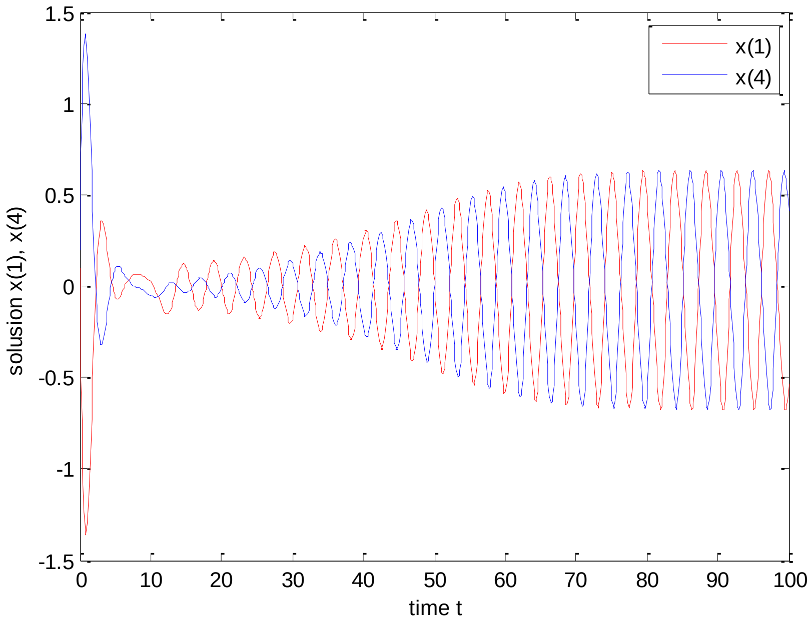

Fig2. *A branch of anti-phased periodic solution is bifurcated from the trivial solution with* $\tau = 0.5$.

ACKNOWLEDGEMENTS

This research was supported by the Natural Science Foundations of Heilongjiang Province (A2015016) .

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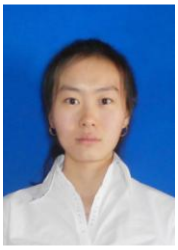
AUTHORS' BIOGRAPHY

Chunrui Zhang, received her PhD degree in mathematics from Harbin Institute of Technology. She is a professor of the Department of Mathematics, Northeast Forestry University. Her current research interests include Differential Equation, Dynamical Systems and Mathematical Biology.

Hongpeng Li, He is a Ph.D student of the college of Mechanical and electrical engineering, Northeast Forestry University. His main research direction is Differential Equation and Dynamical Systems.



Ye Sun, She is a Master student of the Department of Mathematics, Northeast Forestry University. Her current research interests include Differential Equation, Dynamical Systems and Mathematical Biology.