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On Groups with Chain Conditions on Subnormal Subgroups

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Abstract: *Groups with chain Conditions on subnormal subgroups have been investigated by many authors. In this paper we give a necessarily and sufficient conditions under which a group G satisfy the ascending or the descending chain conditions on subnormal subgroups.*

Keywords: *Prime Subgroups, Groups with chain Conditions on subnormal subgroups.*

1. Introduction

Let G be a group. A subgroup P of G is said to be a *primesubgroup* of G if P is normal in G and $[A,B] \subseteq P$ with $A,B \lhd G$ implies that either $A \subseteq P$ or $B \subseteq P$. Here [,] is the commutator. Following Scukin [11] we say that a group G is *prime* if $[A,B] \neq 1$ whenever A and$B$ are nontrivial normal subgroups of G, see also Dark [5].Then P is prime in G if and only if G/P is a prime group.

We define the *soluble radical* $\sigma(G)$ to be the product of all soluble normal subgroups of G. We say that G is *semisimple* if $\sigma(G)$ =1.The terms of the *derived* and *lower central series* of G are denoted $G^{(n)}$ and $\gamma_n(G)$ as in Robinson [8]. A prime subgroup P of G is said to be a *minimal prime subgroup belonging to a normal subgroupH* if $P \supseteq H$ and if there is no prime subgroup between H and P,seeKurata [6, p 205]. The *radical*r(H) of a normal subgroup in G is the intersection of all minimal prime subgroups belonging to H, see Kurata [6, p 206]. If G is unclear we write this as $r_G(H)$. It follows that $r(H)$ is the intersection of all prime subgroups containing H, see Kurata [6, Proposition 1.13 p.207]. We write r_G. for $r_G(1)$., the intersection of all minimal prime subgroups of G.

We denote by $Max-\lhd$ the class of all groups satisfying the maximal condition on normal subgroups (often called Max-n), with similar definition for $Min-\lhd, Max-\lhd^n$, $Min-\lhd^n$.The classes of all groups satisfying the maximal (respectively minimal) condition on subnormal subgroups are denoted by Max-sn, and Min-sn, following Robinson[8],which is also our source for any other unexplained notation and determined by the corresponding chain condition, so that G satisfies Min-sn and $G \in Min-sn$ are equivalent statement.

2. Result

Proposition 1: For all group G,

(a) $\sigma(G) \subseteq r_G$.

(b) If $G \in Max-\lhd$,then $\sigma(G) = r_G$.

Proof

(a) Let H be a soluble normal subgroup of G.Then $H^{(n)} = 1$ for some $n \geq 0$. In particular $H^{(n)} \subseteq P$ for every prime subgroup P. Inductively we see that $H \subseteq P$,whence $\sigma(G) \subseteq r_G$.

(b) Let $R = r_G$ and suppose that R not soluble. Let C be the collection of all normal subgroups N of G such that $R^{(n)} \not\subset N$ for all integers $n \geq 0$.Then C is non- empty since $1 \in C$.Hence C has a maximal element say p. We claim that P is prime .Suppose not, then there are normal subgroups A, B of G such that $A \not\subset P$ and $B \not\subset P$ but $[A,B] \subseteq P$.Therefore $AP, BP \notin C$.Hence $R^{(n)} \subseteq AP$ and $R^{(m)} \subseteq BP$ for some integers $m,n \geq 0$. Let s=max{m,n}. Then $R^{(s+1)} \subseteq [AP,BP] = [AP,B][AP,P] = [A,B][P,B][A,P][P,P] \subseteq P$. Hence $AP \subseteq P$ or $BP \subseteq P$, which implies that $A \subseteq P$ or $B \subseteq P$, a contradiction. Hence P is prime and $R \not\subset P$, another contradiction. Therefore R is soluble so $R \subseteq \sigma(G)$. But $\sigma(G) \subseteq R$ by(a),so $R = \sigma(G)$ as claimed.

Proposition 2

(a) Let $G \in Max-\triangleleft^3$.Then $\sigma(G)$ is soluble and $\sigma(G) \in Max$.

(b) Let $G \in Min-\triangleleft^2$. Then $\sigma(G)$ is soluble and $\sigma(G) \in Min$.

Proof:

(a) Since in particular $G \in Max-\triangleleft$ it follows that $S = \sigma(G)$ is the product of finitely many soluble normal subgroups, hence is soluble. Because $G \in Max-\triangleleft^3$ we have $S \in Max-\triangleleft^2$. Each derived factor $S^{(n)}/S^{(n+1)}$ is abelian with $Max-\triangleleft$, hence with Max. By E-closure of Max, we have $S \in Max-\triangleleft$.

(b)By Theorem 5.49.1 or Robinson [8]p. 148 we have $Min-\triangleleft^2 = Min-sn$.Now apply the analogous argument to part (a) with Max replaced by Min.

Proposition 3 Let G be a group and S be respectively the set of normal subgroups, subnormal subgroups, n-step subnormal subgroups of G. Suppose that $N_i \triangleleft G$ (i=1,…,m) and $\bigcap_{i=1}^{m} N_i = 1$.

Let $S_i = \{HN_i/N_i : H \in S\} \in Max-S_i$ (respectively $Min-S_i$) for all I,

then $G \in Max-S$ (respectively $Min-S$).

Proof: This is equivalent to R_0-closure of these classes, see Robinson[8] Corollary to Lemma 1.48,p.39.

Proposition 4 A group G is a sub-direct product of a family of groups $\{G_\alpha\}_{\alpha \in A}$ if and only if for each $\alpha \in$ A there is a surjective homomorphism $g_\alpha : G \to G_\alpha$ such that $\bigcap_{\alpha \in A} \ker g_\alpha = 1$.

Proof: This is standard: compare Cohn [3, p.99]

Corollary 5 Let G be a group and let $G_\alpha\}_{\alpha \in A}$ be a family of normal subgroups of G.

If $\bigcap_{\alpha \in A} G_\alpha = 1$, then G is a sub-direct product of the family of groups $\{G/G_\alpha\}_{\alpha \in A}$

Proposition 6

(a) G is semi-simple with $Max-\triangleleft^n$ ($n \geq 1$) (respectively $Max-sn$) if and only if G is a sub-direct product of a finite number of prime groups satisfying $Max-\triangleleft^n$ ($n \geq 1$) (respectively $Max-sn$).

(b) G is semi-simple with $Min-\triangleleft^n$ ($n \geq 1$) (respectively $Min-sn$) if and only if G is a sub-direct product of a finite number of prime groups satisfying $Min-\triangleleft^n$ ($n \geq 1$) (respectively $Min-sn$).

Proof: (a) Let G be semisimple with $Max-\lhd^n$ ($n \geq 1$) (respectively $Max-sn$).Then $\sigma(G)$ =1.By Kurata [3] Proposition 4p 214 we have $r_G = \bigcap_{i=1}^{m} P_i$ where the P_i are minimal prime subgroups of G. But by Proposition 1(b) $\sigma(G) = r_G$,so $\sigma(G)$ =1.Since P_i is a prime subgroup the quotient G/P_i is prime, and by Q-closure it lies in $Max-\lhd^n$ ($n \geq 1$) (respectively $Max-sn$) .By Corollary 5 G is a subdirect product of prime groups satisfying $Max-\lhd^n$ (respectively $Max-sn$).

To prove the converse suppose that G is a subdirect product of finitely many prime groups G_i where i=1,…,m and each G_i satisfies $Max-\lhd^n$ (respectively $Max-sn$).Let $g_i : G \to G_i$ be the homomorphism of Proposition 4. For each I we have $G/\ker g_i \cong G_i$, and G_i is prime. So $\ker g_i$ is a prime subgroup of G. Thus $r_G \subseteq \ker g_i$ for all i, so $r_G = 1$. By proposition 1(a) also $\sigma(G)$ =1, so G is semisimple. That $G \in Max-\lhd^n$ (respectively $Max-sn$) follows from Proposition 3.

(b) Let G be semisimple with $Min-\lhd^n$ ($n \geq 1$) (respectively $Min-sn$) .Then G is has only a finite number of minimal normal subgroups M_i where i=1,…r. Let P_i be a normal subgroup of G that is maximal with respect to not containing M_i. We claim that P_i is a prime subgroup of G. If not there exist normal subgroups A,B of G such that $A \not\subset P_i$, $B \not\subset P_i$, but [A,B]$\subseteq P_i$. Now $P_i \subset AP_i$ and $P_i \subset BP_i$, so by the choice of P_i we have $AP_i \supseteq M_i$ and $BP_i \supseteq M_i$. Therefore $\gamma_2 M_i \subseteq [AP_i, BP_i] \subseteq P_i$.But $\gamma_2 M_i \neq 1$ since G is semi-simple so $\gamma_2 M_i = M_i \subseteq P_i$.Therefore P_i is a prime subgroup of G and G/P_i is a prime group. If $\bigcap_{i=1}^{m} P_i \neq 1$,then this intersection contains some minimal subgroup M_j .But $M_j \not\subset P_j$, a contradiction. Therefore $\bigcap_{i=1}^{m} P_i = 1$ and Corollary 5 implies that G is a sub-direct product of a finite number of prime groups with $Min-\lhd^n$ (respectively $Min-sn$) .The converse is as in part(a).

We now come to our main theorem:

Theorem 7 Let G be a group. Then

(a) $G \in Max-\lhd^n$ $(n \geq 3)$(*respectively Max-sn*) if and only if
(i) $\sigma(G)$ is soluble with Max.
(ii) $G/\sigma(G)$ is a sub-direct product of finitely many prime groups satisfying

$Max-\lhd^n$ $(n \geq 3)$(*respectivelyMax* $- sn$)

(b) $G \in Min-\lhd^n$ $(n \geq 2)$(*respectively* (or equivalently *Min-sn*) if and only if

(i) $\sigma(G)$ is soluble with Min.

(ii) $G/\sigma(G)$ is a sub-direct product of finitely many prime groups satisfying

$Min-\lhd^n$ $(n \geq 2)$(*respectivelyMin* $- sn$)

Proof: Combine Propositions 2 and 6.

Corollary 8: G is a finite group if and only if $\sigma(G)$ is finite and $G/\sigma(G)$ is a subdirect product of finitely many finite prime groups.

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