International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 3, Issue 1, January 2015, PP 18-24
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Application of Optimal q-Homotopy Analysis Method to Second Order Initial and Boundary Value Problems

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Abstract: *In this study the application of a newly developed efficient method namely, optimal q-homotopy analysis method (Oq-HAM) has been illustrated for solving second order initial and boundary value problems. The Oq-HAM is a flexible method and can be applied to solve different types of problems. Moreover, it can easily be implemented in symbolic soft computing tools, e.g. MATHEMATICA.*

Keywords: *Optimal q-homotopy analysis method, Second order initial\boundary value problems.*

1. Introduction

The homotopy analysis method (HAM) is an analytic method that provides series solutions and has been proposed first by Liao [1]. It has been successfully applied to linear and non-linear equations in various fields of engineering and science [2-12]. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called h-curve, it is easy to determine the valid regions of h to gain a convergent series solution. El-Tawil and Huseen [13] established a method namely q-homotopy analysis method (q-HAM) which is a more general method of HAM, the q-HAM contains an auxiliary parameter n as well as h such that the case of n=1 (q-HAM; n=1) the standard homotopy analysis method (HAM) can be reached. The q-HAM has been successfully applied to numerous problems in science and engineering [13-19]. Recently, Huseen et al. [20] have introduced and developed a new method, called optimal q-homotopy analysis method (Oq-HAM). An advantage of Oq-HAM over the HAM and q-HAM is that there is no necessity to identify the h-curve. Our goal of this paper is to apply the Oq-HAM introduced by Huseen et al. [20] for solving second order initial and boundary value problems.

2. Basic Idia of the Optimal q-Homotopy Analysis Method (Oq-HAM).

Consider the following differential equation

$$N[u(t)] = 0, \tag{1}$$

where N is a nonlinear operator, $u(t)$ is an unknown function.

Let us construct the so-called zeroth-order deformation equation

$$(1-nq)L[\emptyset(t;q) - u_0(t)] = F(n)qN[\emptyset(t;q)], \tag{2}$$

where $F(n)$ is a nonzero auxiliary function, $n \geq 1$, $q \in [0, \frac{1}{n}]$ denotes the so-called embedded parameter, L is an auxiliary linear operator. Choosing the function $F(n)$ depends on the given problem. It is obvious that when $q = 0$ *and* $q = \frac{1}{n}$ equation (2) becomes

$$\emptyset(t;0) = u_0(t), \quad \emptyset\left(t;\frac{1}{n}\right) = u(t) \tag{3}$$

Respectively. Thus as q increases from 0 to $\frac{1}{n}$, the solution $\emptyset(t;q)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$.

Having the freedom to choose $u_0(t), L, F(n)$ we can assume that all of them can be properly chosen so that the solution $\emptyset(t;q)$ of equation (2) exists for $q \in [0, \frac{1}{n}]$.

Expanding $\emptyset(t;q)$ in Taylor series, one has

$$\emptyset(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) q^m , \tag{4}$$

where

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \emptyset(t;q)}{\partial q^m}_{q=0} \tag{5}$$

Assume that $F(n), u_0(t), L$ are so properly chosen such that the series (4) converges at $q = \frac{1}{n}$ and

$$u(t) = \emptyset\left(t; \frac{1}{n}\right) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) \left(\frac{1}{n}\right)^m \tag{6}$$

Defining the vector $\vec{u}_r(t) = \{u_0(t), u_1(t), u_2(t), \ldots, u_r(t)\}$. Differentiating equation (2) m times with respect to q and then setting $q = 0$ and finally dividing them by $m!$ we have the so-called m^{th} order deformation equation

$$L[u_m(t) - k_m u_{m-1}(t)] = F(n) R_m(\vec{u}_{m-1}(t)), \tag{7}$$

where

$$R_m(\vec{u}_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\emptyset(t;q)]}{\partial q^{m-1}}_{q=0} \tag{8}$$

and

$$k_m = \begin{cases} 0 & m \leq 1 \\ n & otherwise \end{cases} \tag{9}$$

It should be emphasized that $u_m(t)$ for $m \geq 1$ is governed by the linear equation (7) with linear boundary conditions that come from the original problem. Let

$$\Delta_m = \int_\Omega (N[U_m(t)])^2 d\Omega , \tag{10}$$

where

$$U_m(t) = u_0(t) + \sum_{i=1}^{m} u_i(t) \left(\frac{1}{n}\right)^i ,$$

denote the square residual error of the mth-order approximation of the equation (1) integrated in the whole domain Ω, In theory if the square residual error Δ_m tends to zero, then

$$U_m(t) = \sum_{i=0}^{m} u_i(t) \left(\frac{1}{n}\right)^i ,$$

is a series solution of the original equation (1). Besides, at the given order of approximation, the minimum of the squared residual error Δ_m corresponds to the optimal approximation, hence the optimal value of the convergence-control parameter n that corresponds to the minimum of Δ_m.

In this paper, the command NMinimize of the computer algebra system Mathematica is used to find out the minimum of squared residual and the corresponding optimal convergence-control parameter.

3. Numerical Examples

Example 1: Consider the second order nonlinear boundary value problem [21]

$$u''(x) = 2\, u(x)\, u'(x), \qquad 0 \leq x \leq b < \frac{\pi}{2}, \tag{11}$$

with the boundary conditions

$u(0) = 0, \quad u'(b) = sec^2(b),$

where b is the interval length

The exact solution is $u(x) = \tan(x)$.

We choose auxiliary linear operator

$$L[\emptyset(x;q)] = \frac{\partial^2 \emptyset(x;q)}{\partial x^2}$$

With the property

$$L[c_0 + c_1 x] = 0.$$

where c_0, c_1 are integral constants.

We define a nonlinear operator as

$$N[\emptyset(x;q)] = \frac{\partial^2 \emptyset(x;q)}{\partial x^2} - 2\emptyset(x;q)\frac{\partial \emptyset(x,t;q)}{\partial x}.$$

We choose the initial approximation $u_0(x) = x$

According to the zeroth-order deformation equation (2) and the mth-order deformation equation (7) with

$$R_m\left(\vec{u}_{m-1}(x)\right) = \frac{d^2 u_{m-1}}{dx^2} - 2\sum_{i=0}^{m-1} u_i \frac{du_{m-1-i}}{dx}$$

The solution of the mth-order deformation equation (7) for $m \geq 1$ becomes

$$u_m(x) = k_m u_{m-1}(x) + F(n)L^{-1}\left[R_m\left(\vec{u}_{m-1}(x)\right)\right].$$

with the boundary conditions

$$u_m(0) = 0\,, u_m'(0) = 0$$

Let $F(n) = n - n^2, \quad n > 1$

We now successively obtain

$$u_1 = -\frac{1}{3}(1-n)nx^3$$

$$u_2 = -\frac{1}{3}(1-n)n^2x^3 - \frac{1}{15}(1-n)(-1+n)n^2x^3(-5+2x^2)$$

$u_m(x,n)\,,(m = 3,4,\dots)$ can be calculated similarly.Then the series solution expression by Oq-HAM can be written in the form

$$u(x,n) \cong U_m(x,n) = \sum_{i=0}^{m} u_i(x,n)\left(\frac{1}{n}\right)^i \qquad (12)$$

Equation (12) is a family of approximation solutions to the problem (11) in terms of the convergence-control parameter n.

It is found that

$$\Delta_1 = \frac{11092}{4455} - \frac{19192n}{10395} + \frac{8284n^2}{10395} - \frac{920n^3}{6237} + \frac{4n^4}{99}$$

$$\Delta_2 = \frac{20687348468}{3273645375} - \frac{42130247072n}{3273645375} + \frac{146439326104n^2}{9820936125} - \frac{38935691744n^3}{3273645375} + \frac{4732352524n^4}{654729075} - \frac{29434194464n^5}{9820936125} + \frac{781014424n^6}{1091215125} - \frac{4371776n^7}{51962625} + \frac{5295556n^8}{1402990875}$$

$\Delta_m(n), (m = 3,4,\dots)$ can be calculated similarly.

The residual error of Oq-HAM shown in table (1). Fig. (1) shows the comparison between U_5 of Oq-HAM and the exact solution. Fig. (2) shows the absolute error of the 5th order solution Oq-HAM approximate calculated by

$$\text{Absolute Error} = \left|u_{exact} - u_{apprax}\right| \tag{13}$$

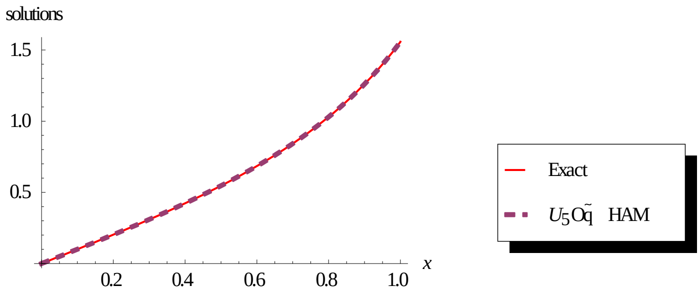

Fig.(1). *Comparison between U_5 of Oq-HAM and the exact solution of problem (11) at $b = 1$.*

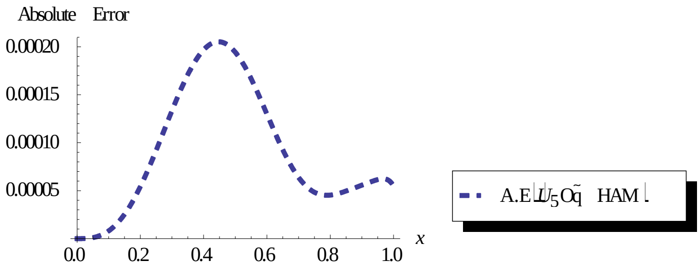

Fig.(2). *The absolute error of U_5 Oq-HAM for problem (11) at $b = 1$.*

Table (1). *The residual of optimal q-homotopy analysis method (Oq-HAM) for problem (11) at $b = 1$.*

Order $\boldsymbol{m}$	n_m	$\Delta_m(OqHAM)$
1	1.42954	1.21686
2	2.72797	0.182938
3	2.57366	0.717682e-2
4	2.51678	0.423612e-3
5	2.48082	0.319238e-4

Example 2: Consider the Klein-Gordon equation

$$u_{tt} - u_{xx} = u \tag{14}$$

Subject to the initial conditions

$u(x,0) = 1 + \sin(x), \quad u_t(x,0) = 0$

With exact solution

$$u(x,t) = \sin(x) + \cosh(t)$$

We choose auxiliary linear operator

$$L[\emptyset(x,t;q)] = \frac{\partial^2 \emptyset(x,t;q)}{\partial t^2},$$

with the property

$$L[C_1(x)t + C_2(x)] = 0,$$

where $Ci\ (i = 1,2)$ are integral constants.

We define a nonlinear operator as

$$N[\emptyset(x,t;q)] = \frac{\partial^2 \emptyset(x,t;q)}{\partial t^2} - \frac{\partial^2 \emptyset(x,t;q)}{\partial x^2} - \emptyset(x,t;q)$$

We choose the initial approximation

$$u_0(x,t) = 1 + \sin(x)$$

According to the zeroth-order deformation equation (2) and the mth-order deformation equation (7) with

$$R_m\left(\vec{u}_{m-1}(x,t)\right) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} - u_{m-1}$$

The solution of the mth-order deformation equation (7) for $m \geq 1$ becomes

$$u_m(x,t) = k_m u_{m-1}(x,t) + F(n)L^{-1}[R_m\left(\vec{u}_{m-1}(x,t)\right)],$$

with the boundary conditions

$$u_m(x,0) = 0 \ , u_{m_t}(x,0) = 0$$

Let $F(n) = n - n^2, \quad n > 1$

We now successively obtain

$$u_1 = -\frac{1}{2}(n - n^2)t^2$$

$$u_2 = -\frac{1}{2}n(n - n^2)t^2 - \frac{1}{24}(-1 + n)n(n - n^2)t^2(-12 + t^2)$$

$u_m(x,t,n)$, $(m = 3,4,\ldots)$ can be calculated similarly.Then the series solution expression by Oq-HAM can be written in the form

$$u(x,t,n) \cong U_m(x,t,n) = \sum_{i=0}^{m} u_i(x,t,n)\left(\frac{1}{n}\right)^i \qquad (15)$$

Equation (15) is a family of approximation solutions to the problem (14) in terms of the convergence-control parameter n .

It is found that

$$\Delta_1 = \frac{64}{15} - \frac{16n}{5} + \frac{14n^2}{15}$$

$$\Delta_2 = \frac{40024}{2835} - \frac{12256n}{405} + \frac{27416n^2}{945} - \frac{36976n^3}{2835} + \frac{6166n^4}{2835}$$

$\Delta_m(n)$, $(m = 3,4,\ldots)$ can be calculated similarly. The residual error of Oq-HAM shown in table (2). Fig. (3) shows the comparison between U_4 of Oq-HAM and the exact solution. Fig. (4) shows the absolute error of the 4th order solution Oq-HAM approximate calculated by (13)

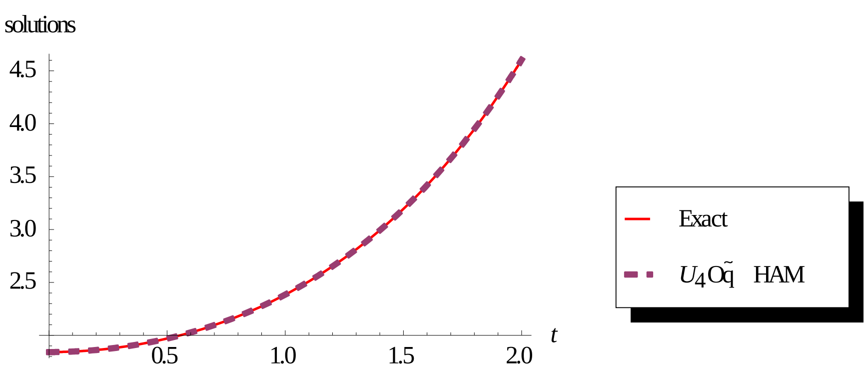

Fig.(3). *Comparison between* U_4 *of Oq-HAM and the exact solution of problem (14) at* $x = 1, 0 \leq t \leq 2$.

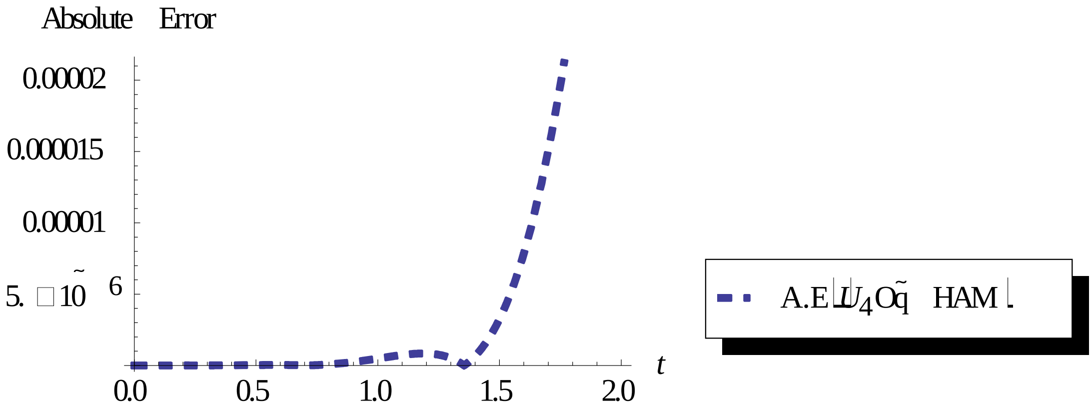

Fig.(4). *The absolute error of* U_4 *Oq-HAM for problem (14) at* $x = 1$ *,*$0 \leq t \leq 2$ *.*

Table (2). *The residual of optimal q-homotopy analysis method (Oq-HAM) for problem (14)*

Order $\boldsymbol{m}$	n_m	$\Delta_m(OqHAM)$
1	1.71429	1.52381
2	2.14212	0.114172e-1
3	2.04637	0.276422e-4
4	2.02191	0.323444e-7

4. Conclusion

The effectiveness of Oq-HAM has been established by solving second order initial and boundary value problems. The results reveal that the Oq-HAM has high accuracy to determine the convergence-control parameter; hence the results match well with the exact solutions and this proves the effectiveness of the method. The Oq-HAM can easily be programmed in symbolic languages as available in standard mathematical soft computing tools e.g. MATHEMATICA.

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