International Journal of Scientific and Innovative Mathematical Research (IJSIMR)
Volume 2, Issue 9, September 2014, PP 782-793
ISSN 2347-307X (Print) & ISSN 2347-3142 (Online)
www.arcjournals.org

Comparison of Godunov's and Relaxation Schemes Approximation of Solutions to the Traffic Flow Equations

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Abstract: *In this paper we deal with the study of the traffic flow model that is governed by two hyperbolic equations. By analysing the equations we obtain two real and distinct eigen values which enables us to determine the wave structure of the possible solution to the Riemann problem set up. We then obtain the numerical solution to the Riemann problem that we set up using the Godunov scheme and the relaxation scheme. Finally, we compare the results obtained from these two schemes graphically and explain in details.*

Keywords: *Eigen values, Riemann problem, Rankine-Hugonoit, Integral curves, Relaxation scheme, Godunov scheme, Weak solution*

1. Introduction

Eigenvalues physically represent speeds of propagation of information. Depending on the initial data the eigenvalues may represent shock and rarefaction waves. A shock wave commonly known as shock is a type of propagating disturbance that is characterized by an abrupt but nearly discontinuous change in characteristic of the medium. Across a shock there is always an extremely rapid rise in pressure, temperature and density of the flow.

Weak solution to an ordinary or partial differential equation is a function for which the derivatives may not all exist but which is nonetheless deemed to satisfy the equation in some precisely defined sense, Leveque[4].

A Riemann problem consists of equations together with the discontinuous initial data. Using local relaxation approximation Shi Jin *et al*[5]. constructed a linear hyperbolic system with a stiff lower order term that approximates the original system with a small dissipative correction. The main feature of this class of schemes was its simplicity and generality since it used neither Riemann solvers spatially nor non-linear systems of algebraic equations solvers temporally, yet it could achieve high order accuracy and picked up the right weak solutions. Also Godunov proposed a way to make use of the characteristic information within the framework of a conservative method by suggesting solving the Riemann problems forward in time and the solutions were easy to compute as well as gave substantial information about the characteristic structure and lead to conservation methods since they were themselves exact solutions of the conservation laws and hence conservative, Leveque[4].

2. Mathematical Formulation of the Traffic Flow Equations

Using the conservative form of the AW-Rascle model we set up the Riemann problem (2.1) with piecewise constant initial data (2.2)

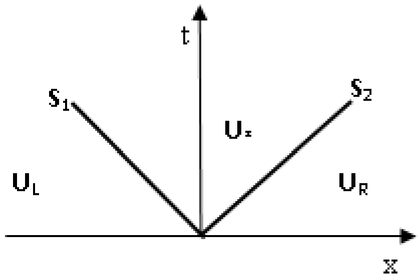
$$U_t + F(U)_x = 0 \tag{2.1}$$

With initial conditions

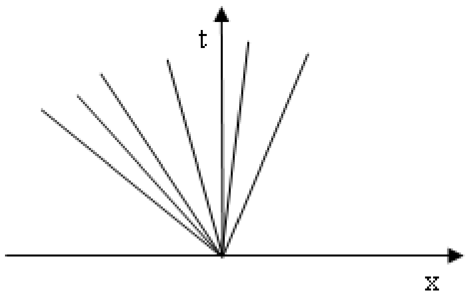
$$U(x,0) = \begin{cases} U_L & x \le 0 \\ U_R & x > 0 \end{cases} \tag{2.2}$$

Where $U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \rho \\ y \end{bmatrix}$ $F(U) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \rho u \\ yu \end{bmatrix}$ for $y = \rho u + \rho p(\rho)$

The general solution of the Riemann problem includes a 1-wave connecting to the left state U_L to an intermediate state U_* (to be defined) and a 2-wave connecting this intermediate state to the right state U_R. Since the 1-waves can either be shocks or rarefaction waves there will be the following types of solutions:



- 1-shock of speed S_1, connecting U_L to an intermediate state U_* followed by a 2-contact discontinuity connecting U_* to the right state U_R.



- 1-rarefaction wave, connecting U_L to an intermediate state U_* followed by a 2-contact discontinuity connecting U_* to the right state U_R.

The shock speed S_1 can either be less than zero or greater than zero. To determine the intermediate state we need to compute the Riemann invariants in the sense of Lax and use the *i*-Lax curves to represent the solution of our preferred eigen structure, for *i=1,2* Kimathi *et al*.[3]. Thus to determine the *i*-Lax curves, we arbitrary choose the situation where a given left state U_L can be connected to an arbitrary state U_* on the right by a 1-shock of speed S_1. Thus for any discontinuity of speed S_1 to satisfy the Rankine-Hugoniot condition we write:

$$\begin{aligned} \rho_* u_* - \rho_L u_L &= S_1(\rho_* - \rho_L) \\ y_* u_* - y_L u_L &= S_1(y_* - y_L) \end{aligned} \tag{2.3}$$

Where on eliminating S1 and simplifying we have:

$$\frac{y_*}{\rho_*} = \frac{y_L}{\rho_L} \tag{2.4}$$

And since state U_* is arbitrary the *i*-Lax curve passing through U_L are obtained from equation (2.4) in terms of the primitive variables as Kimathi *et al*.[3]

$$L_1(\rho;\rho_L,u_L)=u_L+p(\rho_L)-p(\rho)$$

Similarly, the 2-Lax curves are obtained by considering the Riemann invariants $L_2(\rho;\rho_R,u_R)=u_R$

Now stating the Riemann invariants W_1 and W_2 associated with the respective characteristics λ_1 and λ_2 we have:

$$W_1=u+p(\rho) \qquad W_2=u$$

3. Numerical Solutions

3.1. Relaxation Scheme

Let's consider systems of conservation laws in one dimension (2.1).

Now introducing the relaxation system corresponding to the above equation, we have

$$\begin{aligned} &U_t+V_x=0 \\ &V_t+AU_x=-\frac{1}{\xi}(V-F(U)) \end{aligned} \tag{3.1}$$

Where $\xi>0$ and;

$$V=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix},\quad A=\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

The matrix A is a positive diagonal matrix to be chosen.

For ξ sufficiently small, it is expected that by solving (3.1) properly, one can obtain good approximations to the original conservation laws

The positive constant a need satisfy: Shi J. *et al.* [5]

$$-\sqrt{a}\le F'(U)\le\sqrt{a} \qquad \forall U$$

For the relaxation system (3.1), the initial data is:

$$\begin{aligned} U(x,0)&=U_0(x) \\ V(x,0)&=V_0(x)\equiv F(U_0(x)) \end{aligned}$$

Now introducing the spatial grid x_j with mesh width $h_j=x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}$ while the discrete time level t_n with time step $k=t_{n+1}-t_n$ for n=0, 1, 2.

A spatial discretization to equation (3.1) in conservation form can be written as:

$$\begin{aligned} &\frac{\partial}{\partial t}U_j+\frac{1}{h_j}(V_{j+\frac{1}{2}}-V_{j-\frac{1}{2}})=0 \\ &\frac{\partial}{\partial t}V_j+\frac{1}{h_j}A(U_{j+\frac{1}{2}}-U_{j-\frac{1}{2}})=-\frac{1}{\xi}(V_j-F_j) \end{aligned} \tag{3.2}$$

Where the averaged quantity F_j is defined by

$$F_j=\frac{1}{h_j}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}F(U)\,dx\approx F(U_j)$$

The relaxation system 3.1 has two characteristic variables, see Shi J. *et al.* [5]

$$v_p \pm \sqrt{A} u_p\,, \quad p = 1\,, 2 \tag{3.3}$$

That travel with the frozen characteristic speeds $\pm\sqrt{A}$ respectively.
For better accuracy we use a second-order scheme that is the Van Leer's MUSCL scheme, see Van Leer[7].

Applying this scheme to the pth component that is equation 3.3, gives

$$\begin{aligned}(v_p + \sqrt{a_p} u_p)_{j+\frac{1}{2}} &= (v_p + \sqrt{a_p} u_p)_j + \frac{1}{2} h_j \sigma_j^+ \\ (v_p - \sqrt{a_p} u_p)_{j+\frac{1}{2}} &= (v_p - \sqrt{a_p} u_p)_{j+1} - \frac{1}{2} h_j \sigma_{j+1}^-\end{aligned} \tag{3.4}$$

Where σ_j is the slope of $v_p \pm \sqrt{a_p} u_p$ on the j-th cell which we define using Sweby's notation, see Sweby[6] .

$$\sigma_j^{\pm} = \frac{1}{h_j}\begin{bmatrix}(v_p)_{j+1} \pm \sqrt{a_p}(u_p)_{j+1} - (v_p)_j \\ \pm \sqrt{a_p}(u_p)_j\end{bmatrix}\phi(\theta_j^{\pm})$$

$$\theta_j^{\pm} = \frac{(v_p)_j \pm \sqrt{a_p}(u_p)_j - (v_p)_{j-1} \pm \sqrt{a_p}(u_p)_{j-1}}{(v_p)_{j+1} \pm \sqrt{a_p}(u_p)_{j+1} - (v_p)_j \pm \sqrt{a_p}(u_p)_j}$$

Where ϕ is a slope-limiter function given as, Van Leer[7]

$$\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}$$

Solving equations 3.4 for $u_{j+1/2}$ and $v_{j+1/2}$ gives

$$\begin{aligned}U_{j+\frac{1}{2}} &= \frac{1}{2}(U_j + U_{j+1}) - \frac{1}{2}\sqrt{a_p}(V_{j+1} - V_j) \\ &+ \frac{1}{4}\sqrt{a_p}(h_j \sigma_j^+ + h_{j+1}\sigma_{j+1}^- \\ V_{j+\frac{1}{2}} &= \frac{1}{2}(V_j + V_{j+1}) - \frac{1}{2}\sqrt{a_p}(U_{j+1} - U_j) \\ &+ \frac{1}{4}\sqrt{a_p}(h_j \sigma_j^+ + h_{j+1}\sigma_{j+1}^-\end{aligned} \tag{3.5}$$

Applying 3.5 in 3.2 we have,

$$\begin{aligned}&\frac{\partial}{\partial t}U + \frac{1}{2h_j}(V_{j+1} - V_{j-1}) - \frac{\sqrt{a_p}}{2h_j}(U_{j+1} - 2U_j \\ &+ U_{j-1}) - \frac{1}{4h_j}(h_{j+1}\sigma_{j+1}^- - h_j(\sigma_j^+ + \sigma_{j-1}^-) + h_{j-1}\sigma_{j-1}^+ = 0 \\ &\partial/\partial t\, U + \frac{1}{2h_j}(U_{j+1} - U_{j-1}) - \frac{\sqrt{a_p}}{2h_j}(V_{j+1} - 2V_j \\ &+ V_{j-1}) - \frac{1}{4h_j}(h_{j+1}\sigma_{j+1}^- + h_j(\sigma_j^+ + \sigma_{j-1}^-) - h_{j-1}\sigma_{j-1}^+ \\ &= -\frac{1}{\xi}(V_j - F_{(p)}(U_j))\end{aligned} \tag{3.6}$$

Where $F_{(p)}$ is the p-th component of the flux vector? F

Since the one dimensional systems of equation 3.1 has two eigen values $u + p'(\rho), \quad and\ u$ we take

$$a_1 = \sup|u + p'(\rho)|, \ a_2 = \sup|u|,$$

Denoting $D^{+}W_j = \frac{1}{h_j}(W_{j+\frac{1}{2}} - W_{j-\frac{1}{2}})$

Where W_j^n is the approximate cell average of a quantity W in the cell $\left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]$ at time t_n and by $W^n_{j+\frac{1}{2}}$ the approximate point value of W at $x = x_{j+\frac{1}{2}}$ and $t = t_n$.

Now, to obtain the time discretization for the relaxation scheme we use a second-order TVD Runge-Kutta splitting scheme which was introduced by Jin, Shi J. *et al.* [5]

Second-order TVD Runge-Kutta splitting scheme to the time derivative in 3.6, yields

$$U^{*} = U^{n},$$

$$V^{*} = V^{n} + \tfrac{k}{\xi}(V^{*} - F(U^{*}));$$

$$U^{(1)} = U^{*} - kD^{+}V^{*},$$

$$V^{(1)} = V^{*} - kAD^{+}U^{*};$$

$$U^{**} = U^{(1)},$$

$$V^{**} = V^{(1)} - \tfrac{k}{\xi}(V^{**} - F(U^{**})) - 2\tfrac{k}{\xi}(V^{*} - F(U^{*}))\ U^{(2)} = U^{**} - kD^{+}V^{**},$$

$$V^{(2)} = V^{**} - kAD^{+}U^{**};$$

$$U^{n+1} = \tfrac{1}{2}(U^{n} + U^{(2)}),$$

$$V^{n+1} = \tfrac{1}{2}(V^{n} + V^{(2)});$$

3.2. Godunov Scheme

Considering the Riemann problem, equation 2.1, with:

Initial condition:

$$U(x,0) = U_0(x) = \begin{cases} \tilde{U}_i^n & x < 0 \\ \tilde{U}_{i+1}^n & x > 0 \end{cases}$$

Now we discretize the spatial domain into finite volumes $I_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ of regular size.

Now the data at time level n may be seen as pairs of constant states $\left(U_i^n, U_{i+1}^n\right)$ separated by a discontinuity at the inter cell boundary $x_{i+\frac{1}{2}}$.In order to admit discontinuous solution, we must use one of the integral forms of the conservation laws Toro[1], that is,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}(x,\Delta t)dx = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}(x,0)dx +$$

$$\int_0^{\Delta t} F(\widetilde{U}(x_{i-\frac{1}{2}},t))dt - \int_0^{\Delta t} F(\widetilde{U}(x_{i+\frac{1}{2}},t))dt \qquad (3.7)$$

For the control volume $\left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]\times[0,\Delta t]$ in the domain of interest, Godunov assumes a piecewise constant distribution of the data. This is by defining cell averages which produces the desired piecewise constant distribution, that is,

$$U(x,t^n) = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{U}(x,t^n)dx \qquad (3.8)$$

The time step Δt must be chosen sufficiently small, to avoid wave interactions within a cell. Thus;

$$\Delta t \le \frac{0.75\Delta x}{S_{\max}^n} \qquad (3.9)$$

Where $S_{\max}^n$ is the maximum velocity present throughout the domain at time t^n

Equation (3.9) allows the interaction of waves from the neighboring Riemann problem during the time step provided the interaction is entirely contained within a mesh cell.

The integrand in (3.8) is an exact solution to the conservation laws and thus applying the integral form to the control volume we can show that $U_{i+\frac{1}{2}}(0)$ is the solution to the Riemann problem $\left(U_i^n, U_{i+1}^n\right)$ along the t-axis in the local domain. Similarly $U_{i-\frac{1}{2}}(0)$ is the solution to the Riemann problem $\left(U_{i-1}^n, U_i^n\right)$ along the t-axis, Toro[1]

Thus using equation 3.7, Godunov method can be written in conservation form as

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}\left[F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}\right] \qquad (3.10)$$

With the inter cell numerical flux given by

$$F_{i+\frac{1}{2}}^n = F(U_{i+\frac{1}{2}}(0^-; U_i^n, U_{i+1}^n)) \qquad (3.11)$$

4. SIMULATION

4.1. Relaxation Scheme

We simulate the AW-Rascle type traffic model equation 2.1 by choosing the following form of the speed adaptation coefficient, Kimathi *et al*[3], $P(\rho) = C\ln(\frac{\rho}{1-\rho})$., where $C = 0.9$

Now we investigate two traffic flow scenarios that lead to two solutions of interest namely a 1-shock wave followed by a 2-contact and a 1-rarefaction followed by a 2-contact.

Let the traffic under consideration be along the x-axis and beginning at $x = -30$ and ending at $x = 30$ with traffic street lights at $x = 0$. Let the direction of flow of traffic be in the direction of increasing x along the axis. When the light red goes on, the vehicles approaching the street light decrease their velocity which as a result increases the density of the vehicles and thus causing shock wave.

In the first scenario we consider the initial conditions,

$$\begin{aligned} \rho_L &= 0.6, u_L = 0.95 \\ \rho_R &= 0.6, u_R = 0.1 \end{aligned} \tag{4.1}$$

This data gives rise to a 1-shock wave followed by a 2-contact discontinuity as shown by fig 1.1(a) and 1.1(b). Note that there is an abrupt change in color in fig 1.1(b), check the color bar, due to the abrupt change in density as shown in fig 1.1(a). Also in fig 1.2 we show the velocity profile at the same final computational time obtained using the initial data (4.1).

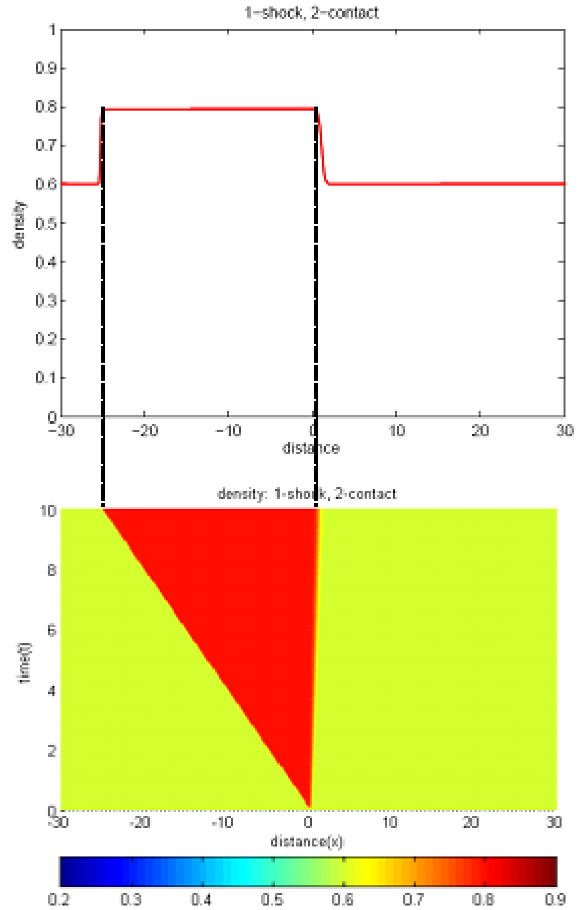


Fig1.1(a). *Density profile and fig 1.1(b): Distance-time graph for the density profile for Relaxation scheme solution to the Riemann problem;*

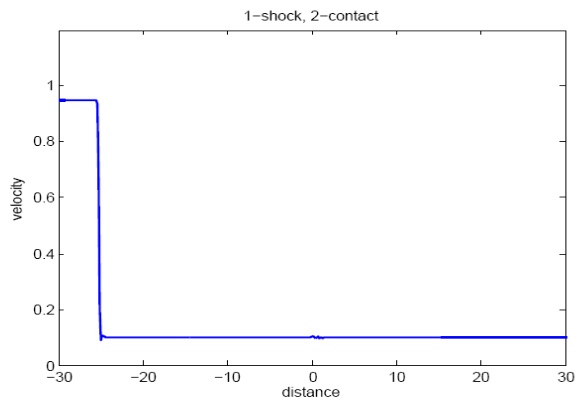


Fig1.2. *Velocity profile for Relaxation scheme solution to the Riemann problem;*

In the second case we consider the initial conditions,

$$\rho_L = 0.6, u_L = 0.1$$
$$\rho_R = 0.6, u_R = 0.95 \qquad (4.2)$$

This data gives rise to a 1-Rarefaction wave followed by a contact discontinuity as shown by fig 1.3(a) and 1.3(b). Fig 1.4 shows the velocity profile at the same final time T=10 obtained using the initial data (4.2)

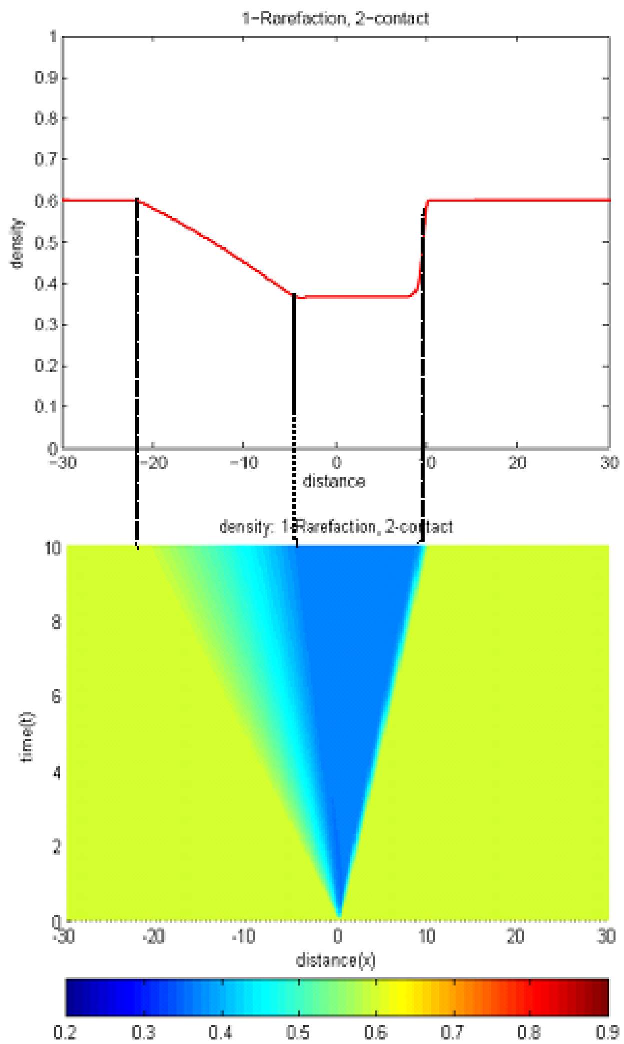


Fig1.3(a). *Density profile and Fig 1.3(b): Distance-time graph for the Density profile for Relaxation scheme solution to the Riemann problem;*

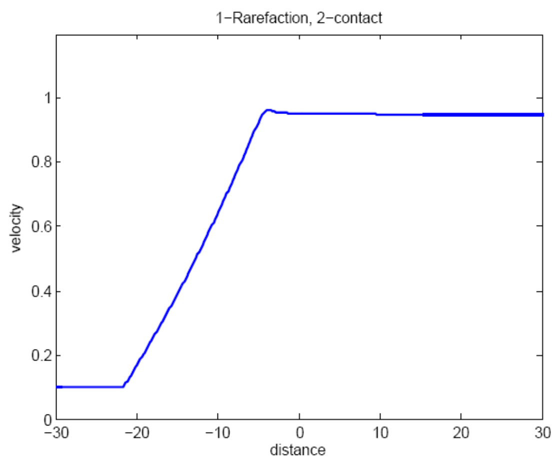


Fig1.4. *Velocity profile for Relaxation scheme solution to the Riemann problem;*

4.2. Godunov Scheme

Similarly, to visualize the features of the Godunov scheme we simulate the AW-Rascle type traffic model equations.

In first scenario we consider the initial conditions (4.1), which give rise to a 1-Shock wave followed by a 2-contact discontinuity as shown in fig 1.5 (a) and 1.5 (b) for the density and velocity profiles respectively.

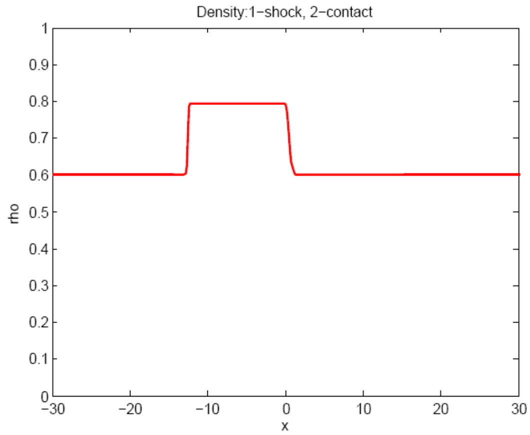


Fig1.5(a). *Density profile for Godunov scheme solution to the Riemann problem;*

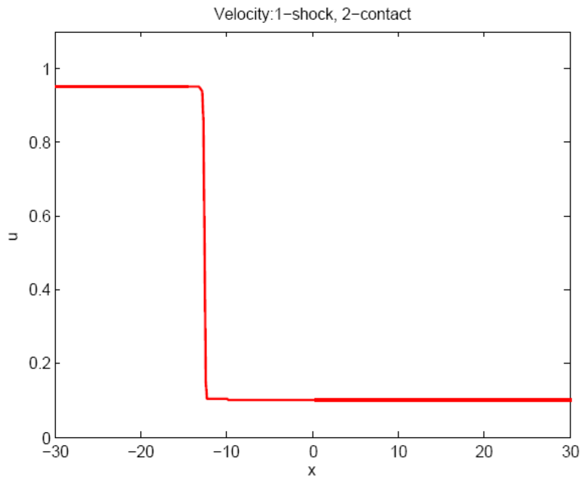


Fig1.5(b). *Velocity profile for Godunov scheme solution to the Riemann problem;*

In the second scenario we consider the initial conditions (4.2), which give rise to a 1-Rarefaction wave followed by a contact discontinuity as shown in fig 1.6 (a) and 1.6 (b) for the density and velocity profiles respectively.

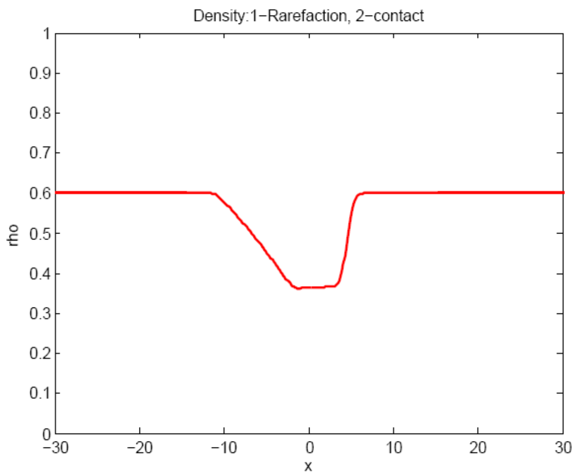


Fig 1.6 (a). *Density profile for Godunov scheme solution to the Riemann problem*

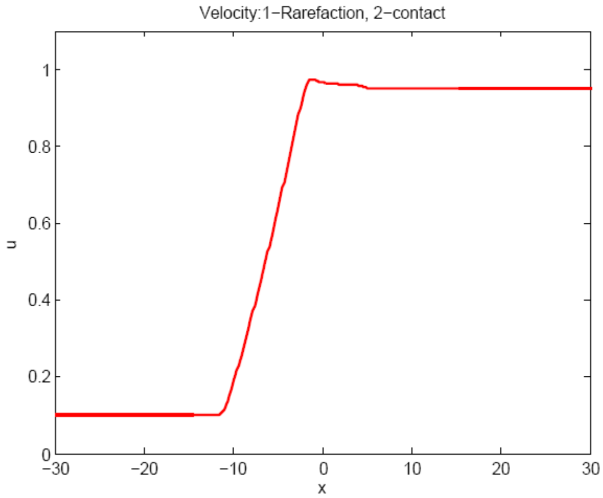


Fig 1.6 (b). *Velocity profile for Godunov scheme solution to the Riemann problem*

5. Comparison

Having considered the velocity and density profiles for the two schemes using the initial conditions 4.1 and 4.2, we now present a comparison for the two schemes as shown in the fig 1.7 (a) and 1.7 (b) below for the first case and 1.8 (a) and 1.8 (b) for the second case.

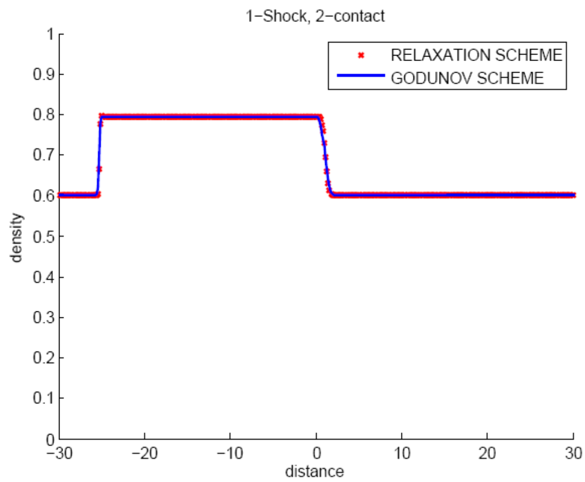


Fig 1.7 (a.). *Density profile comparison for 1-Shock followed by 2-Contact*

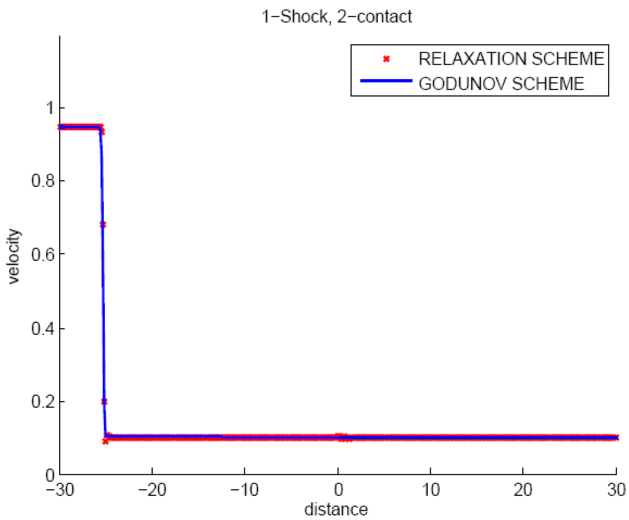


Fig 1.7 (b). *Velocity profile comparison for 1-Shock followed by 2-Contact*

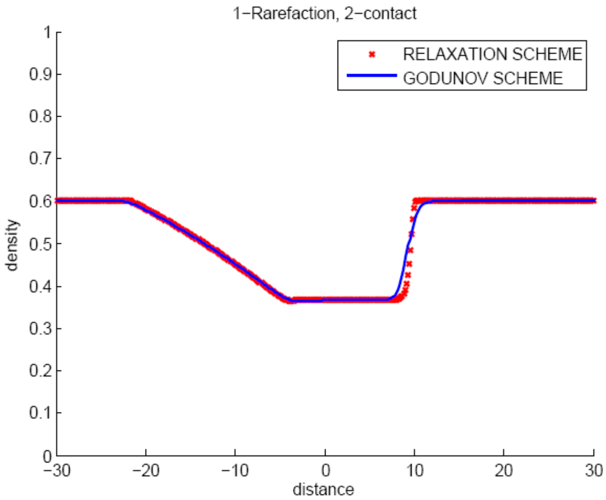


Fig 1.8 (a). *Density profile comparison for 1-Rarefaction followed by 2-Contact*

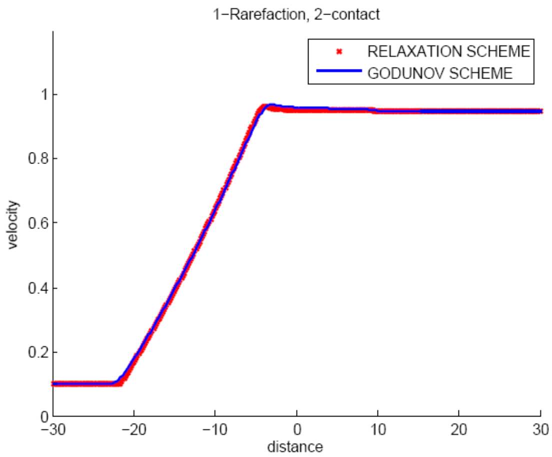


Fig 1.8 (b). *Velocity profile comparison for 1-Rarefaction followed by 2-Contact*

6. Conclusion

From the graphs above it is noted that the relaxation scheme performs equally better as the Godunov scheme. Thus it appears to be more promising and a good alternative to the Godunov scheme because of its simplicity.

Due to insufficient time the authors of this paper decided to consider both numerical schemes that is Godunov scheme and relaxation schemes and thus the authors recommends for comparison of the schemes with the exact solution to check their accuracy so as to show which scheme is more accurate than the other in terms of approximating the exact solutions.

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