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Fixed Point Theorems for a Family of Self-Map on Rings

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Abstract: *A family of self-map* $\{f_i : R \to R \mid i \in N\}$ *on a Ring* $(R, +, \cdot)$*, given by,* $f_i(x) = x + x^i$ *for each x in R is under consideration. In this paper, we will obtain fixed point theorems for these mappings. We will prove that the set of fixed points for these mappings forms ring, Ideal and different Algebraic structures.*

Keywords: *Fixed Point, Ring, Field, Integral Domain, Nilpotent element, Ideal, Nil Ideal.*

1. Introduction

Herstein I. N. in [1] and J. A. Gallian in [2] developed the theory of groups. J. Achari and Neeraj A. Pande in [3] has defined the self-map $f_i : G \to G$ on the group $(G, *)$ as, $f_i(x) = x^i, \forall\ i \in I$ and by finding the fixed points of each f_i he obtained some interesting results of group theory in terms of fixed points of self-maps.

In this paper we consider a family of self- map $\{f_i : R \to R \mid i \in N\}$ on ring $(R, +, \cdot)$, where each f_i *is* given by, $f_i(x) = x + x^i$ for each $x \in R$. Also we will generalize the results for the self-maps f_i defined on ring *R*. Let F_{f_i} denotes the set of fixed points of f_i. In this paper, the discussion about different algebraic structures formed by F_{f_i} under certain conditions is done. We revise some definitions.

Definition 1.1: Let *A* be any set and $f : A \to A$ be a self-map defined on *A*, then an element *a* in *A* is said to be fixed point of *A* if $f(a) = a$.

Definition 1.2: Let $(R, +, \cdot)$, be a commutative ring with unity then an element *x* in *R* is said to be unit in *R* if there exist an element *y* in *R* such that, $x.y = y.x = 1$. In short, units in *R* are the invertible elements of *R*.

Definition 1.3: Let $(R, +, \cdot)$, be a ring then an element *x* in *R* is said to be nilpotent if there exists some positive integer n such that, $x^n = 0$.

Definition 1.4: An Ideal *A* of *R* is said to be nil ideal if every element of *A* is nilpotent element in *R*.

2. Main Results

Theorem 2.1: Let $(R, +, \cdot)$ be a ring and f_i be a self-map on R given by, $f_i(x) = x + x^i$ for each $x \in R$. Then $x \in R$ is fixed point of f_i if and only if *x* is nilpotent element of R with index of nilpotence $k \le i$.

Proof: $x \in R$ is a fixed point of $f_i \Leftrightarrow f_i(x) = x$

$\Leftrightarrow x + x^i = x$

$\Leftrightarrow x^i = 0$

$\Leftrightarrow x \in R$ is nilpotent element of R with index of nilpotence $k \le i$. This completes the proof.

Remark 2.1: Theorem 2.1 immediately suggest that if an element of a ring R is unit then it can't be nilpotent element of R, hence it can't be a fixed point of any of the mappings f_i, for $i \in N$.

Example 2.1: Let $R = \{0, 1, 2, 3, 4, 5\}$ be a ring with respect to the operations addition modulo 6, $\oplus_6$ defined by $a \oplus_6 b = c$ where, c is the least nonnegative integer obtained by dividing $a + b$ by 6, and multiplications modulo 6, $\otimes_6$ defined by $a \otimes_6 b = d$ where d, is the least nonnegative integer obtained by dividing $a.b$ by 6. In R, an element $5 \in R$ is its own inverse. Thus, it is a unit in R. Here, $5^i = 1$ or 5 for all i. Hence $5 \in R$ is not fixed point for any of f_i.

Example 2.2: We know that every non – zero elements of a Field and a Division Ring are units, hence they can't be fixed point for any f_i. Hence, field and division ring has only one fixed point for each f_i, viz. 0. Therefore, we have $F_{f_i} = \{0\}$ for each $i \in N$.

Example 2.3: We know that, an integral domain R has only one nilpotent element which is 0**.** Therefore, for an integral domain R we have $F_{f_i} = \{0\}$, for each $i \in N$.

Example 2.4: Let $(R, +, \cdot)$ be a ring of even integers. Here, R is not an Integral domain as 1 does not belong to R, but R does not have any non - zero nilpotent element. Hence, every f_i, has only one fixed point viz. 0. Therefore we have $F_{f_i} = \{0\}$ for each $i \in N$.

Theorem 2.2: Let $(R, +, \cdot)$ be a ring, consider a self-map f_i on R given by, $f_i(x) = x + x^i$, for each $x \in$ R. Then, $F_{f_i} \neq \emptyset \subseteq$ R, for each i

Proof: We have, 0 is additive identity of R. Therefore, $0^1 = 0$. Thus, 0 is nilpotent element of R with index of nilpotence $1 \le i$ for each i. So, each f_i is guaranteed to have at least one fixed point viz. the additive identity 0 in R. Hence, $F_{f_i} = \{0\}$ for each $i \in N$. Thus, $F_{f_i} \neq \emptyset \subseteq$ R. This completes the proof.

Theorem 2.3: For a ring $(R, +, \cdot)$**,** consider a self-map f_i on R given by, $f_i(x) = x + x^i$, for each $i \in N$. Then $x \in R$ is a fixed point of f_i if and only if $(-x) \in R$ is a fixed point of f_i. Thus, $x \in F_{f_i} \Leftrightarrow -x \in F_{f_i}$

Proof: Let, $x \in F_{f_i}$, *i.e.* x is a fixed point of f_i. Therefore, by Theorem 2.1, we have $x^i = 0$.

Now, $(-x) = (0 - x)$

$$\Rightarrow (-x)^i = (0 - x)^i$$
$$= 0^i - i.0^{i-1}.x + \cdots + (-1)^i.x^i$$
$$= 0$$

$\therefore$ By Theorem 2.1, $(-x)$ is a nilpotent element of R. Thus, $(-x) \in F_{f_i}$

Conversely, suppose $(-x) \in F_{f_i}$ *i.e.* $(-x)$ is a fixed point of f_i. Therefore, by Theorem 2.1, $(-x)^i = 0$,

Now, $(x) = [0 - (-x)]$

$$\Rightarrow (x)^i = [0 - (-x)]^i$$

$$= 0^i - i.0^{i-1}.(-x) + \cdots + (-1)^i.(-x)^i$$
$$= 0$$

Hence, by theorem 2.1, x is a nilpotent element of R. Thus, $x \in F_{f_i}$

This completes the proof.

Example 2.5: Let, R = {0, 1, 2, 3, 4, 5, 6, 7} be a ring with respect to the operations addition modulo 8, $\oplus_8$ defined by $a \oplus_8 b = c$ where, c, is the least nonnegative integer obtained by dividing $a + b$ by 8 and multiplication modulo 8 by $a \otimes_8 b = d$ where, d, is the least nonnegative integer obtained by dividing $a.b$ by 8. Define the mapping $f_4 : R \to R$ as, $f_4(x) = x + x^4$ for each $x \in R$.

Consider, $f_4(2) = 2 + 2^4$
$$= 2 \oplus_8 (2 \otimes_8 2 \otimes_8 2 \otimes_8 2)$$
$$= 2 \oplus_8 0$$
$$= 2$$

Thus, $2 \in R$ is a fixed point of f_4.
We know that, additive inverse of 2 is 6 in R.
Consider, $f_4(6) = 6 + 6^4$
$$= 6 \oplus_8 (6 \otimes_8 6 \otimes_8 6 \otimes_8 6)$$
$$= 6 \oplus_8 0$$
$$= 6$$

Thus, $6 \in R$ is also a fixed point of f_4. Hence, Theorem 2.3 is verified.

Remark 2.2: If $x \in R$ is a fixed point of some f_i , then additive inverse of x is also a fixed point of f_i.

Theorem 2.4: Suppose $(R , +, \cdot)$ be a ring, consider the self-map f_i on R given by

$f_i(x) = x + x^i , \forall\, x \in R$ is a ring homomorphism. If x and y are fixed points of f_i then $x+y$ and $x.y$ are fixed points of f_i.

Proof: Let x and y be any two fixed points of f_i *i.e.* $x, y \in F_{f_i}$. Therefore, $f_i(x) = x$ and $f_i(y) = y$. As f_i is homomorphism, therefore, $f_i(x + y) = f_i(x) + f_i(y) = x + y$. Thus, $x + y$ is a fixed point of f_i . Also, $f_i(x.y) = f_i(x).f_i(y) = x.y$,Thus, $x.y$ is a fixed point of f_i . Hence,$(x + y), (x.y) \in F_{f_i}$. This completes the proof.

Remark 2.3: If we restrict the self-map f_i to be the homomorphism then the set F_{f_i} of fixed points of f_i definitely satisfies the closure property with respect to both the operations. But, if f_i is not homomorphism then the set F_{f_i} may lack the closure property.

Example 2.6: Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in z_2 \right\}$ be a ring with respect to the operations usual addition and multiplication of matrices, where, $(Z_2, \oplus_2 , \otimes_2)$ is the field of integers modulo 2. Define, f_2: $R \to R$ as $f_2(A) = A + A^2$ for each $A \in R$ then,

$F_{f_2} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$. Here, every element of F_{f_2} is its own inverse.

Consider two elements $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in F_{f_2} then,

$$A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } A.\,B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Hence, $A + B$ and $A.B$ are not elements of F_{f_2}.

Thus, closure property with respect to both addition and multiplication is not satisfied.

Thus, we can conclude that F_{f_2} is not a subring of R.

Remark 2.4: Theorem 2.4 gives the sufficient condition for F_{f_i} to satisfy the closure property with respect to both binary operations.

Example 2.7: Let $R = \{0,\ a,\ b,\ c\}$. Define two operations addition, '+' and multiplication '*' on R as,

+	0	A	b	c
0	0	A	b	c
a	a	0	c	b
b	b	C	0	a
c	c	B	a	0

*	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Here, R is a non-commutative ring without unity. Define a self-map f_2 on R as, $f_2(x) = x + x^2$ for each x in R. Here, f_2 is not homomorphism because, $f_2(a+b) = f_2(c) = c$ and $f_2(a) + f_2(b) = 0 + 0 = 0$, but $F_{f_2} = \{0,\ c\}$ is closed under both addition and multiplication defined on R.

Note that, f_1 is not homomorphism for any ring R, but $F_{f_1} = \{0\}$ which is trivial subring of R.

Theorem 2.5: For a ring $(R, +, \cdot)$, suppose a self-map f_i on R given by, $f_i(x) = x + x^i$, for each x in R, is a ring homomorphism., then F_{f_i} itself forms a ring with respect to the operations on R.

Proof: $(R, +, \cdot)$ is a ring and a self-map f_i on R is given by $f_i(x) = x + x^i$, for each x in R is a ring homomorphism, then by Theorem 2.2 F_{f_i} is non-empty subset of R. Now by Theorem 2.4, $x + y$ and $x.y$ are fixed points of f_i, therefore, F_{f_i} satisfies the closure property with respect to both addition and multiplication. Also, the elements of F_{f_i} being the elements of R, satisfies the associative law with respect to both addition and multiplication. By Theorem 2.2, F_{f_i} contains the additive identity 0, thus, $0 \in F_{f_i}$. By Theorem 2.3, F_{f_i} contains the additive inverse of each element. Also the members of F_{f_i} being the members of R satisfies commutative law with respect to addition, also the elements of F_{f_i} satisfies the distributive law as they are elements of R. Hence $(F_{f_i}, +, \cdot)$ is a ring and hence forms subring of R. This completes the proof.

Remark 2.5: If R is a commutative ring then the elements of F_{f_i} being the elements of R satisfies the commutative law, hence each F_{f_i} is commutative.

Theorem 2.6: If $(R, +, \cdot)$ is a commutative ring and if a self-map f_i on *R* given by, $f_i(x) = x + x^i \ \forall\, x \in R$ is homomorphism, then F_{f_i} is an ideal of *R*.

Proof: By Theorem 2.2, F_{f_i} is nonempty subset of *R*, and a self-map f_i on *R* given by, $f_i(x) = x + x^i$ for each *x* in *R* is homomorphism. Consider $x, y \in F_{f_i}$ .Thus we have $f_i(x) = x$, $f_i(y) = y$. By Theorem 2.3, $(-y) \in F_{f_i}$. Therefore, we have $f_i(-y) = -y$. As f_i is a homomorphism, we can write, $f_i(x + (-y)) = f_i(x) + f_i(-y) = x - y$

Hence *x* – *y* is a fixed point of f_i . Thus, $(x - y) \in F_{f_i}$. Hence, F_{f_i} is additive subgroup of *R*.

As *x* is a fixed point of f_i by Theorem 2.1 we have $x^i = 0$. Let *r* be any element of ring *R*. Consider, $(r.x)^i = r^i.x^i = 0$, Thus, by Theorem 2.1, we have $r.x \in F_{f_i}$. Similarly, we can show that $x.r \in F_{f_i}$. Hence F_{f_i} is an ideal of *R*. This completes the proof.

Theorem 2.7: For a Commutative ring $(R, +, \cdot)$, consider a self-map f_i on *R* given by $f_i(x) = x + x^i$ for each $x \in R$ is homomorphism. Then F_{f_i} is Nil ideal of *R*.

Proof: $(R, +, \cdot)$, is a commutative ring and a self-map f_i on *R* defined by $f_i(x) = x + x^i$ for each $x \in R$ is homomorphism. Therefore, by Theorem 2.6 the set F_{f_i} is an ideal of *R*. Suppose *x* be an arbitrary element of F_{f_i} , *i.e. x* is a fixed point of f_i. Hence, by Theorem 2.1 $x^i = 0$. *i.e. x* is nilpotent element of *R*. Thus, every element of F_{f_i} is nilpotent element of *R*. Hence, F_{f_i} is a Nil ideal of *R*. This completes the proof.

Remark 2.6: By Theorem 2.6, we can say that, if *R* is commutative ring and *x* is fixed point of f_i then other fixed points of f_i are easy to find out, as *r.x* is fixed point of f_i for each *r* in *R*. If *R* is a non-commutative ring then, we are interested to find some another fixed points of f_i using one fixed point of f_i. Next theorem helps us to know some another fixed points of f_i.

Theorem 2.8: Let $(R, +, \cdot)$ be any ring and consider a self-map f_i on R given by $f_i(x) = x + x^i$ for each $x \in R$. suppose $x \in R$ is a fixed point of f_i then x^k is also a fixed point of f_i for each $k \in N$.

Proof: $x \in R$ is a fixed point of self-map f_i on *R* defined by, $f_i(x) = x + x^i$ for each $x \in R$. Therefore, by Theorem 2.1, $x^i = 0$ for each $i \in N$. Consider, $(x^k)^i = (x)^{i.k} = (x^i)^k = 0$. Hence, by Theorem 2.1 x^k is also a fixed point of f_i for each $k \in N$.

Theorem 2.9: If $(R, +, \cdot)$ is a ring with unity say 1 and f_i be a self-map on *R* given by, $f_i(x) = x + x^i, \forall\, x \in R$, then $1 \in R$ cannot be a fixed point of any f_i .

Proof: By Theorem 2.1, *x* is a fixed point of f_i if and only if $x^i = 0$. Thus, *x* is a fixed point of f_i if *x* is nilpotent element of *R*. Therefore, 1 is fixed point of f_i if and only if $(1)^i = 0$. But 1 is multiplicative identity in *R*. Hence, $(1)^i = 1 \ \forall\, i \in N$. Thus, $1 \in R$ need not be a fixed point of any f_i . This completes the proof.

Remark 2.7: If *R* is a ring with unity 1, then according to Theorem 2.9, $1 \notin F_{f_i}$, Hence, $F_{f_i} \neq R$, for each *i*. Thus by Theorem 2.2 and Theorem 2.9 each F_{f_i} is proper subset of *R*.

Remark 2.8: According to Theorem 2.5 and Theorem 2.9 if a self-map f_i is homomorphism defined on a ring with unity *R*, and then F_{f_i} is proper subring of *R*.

Remark 2.9: According to Theorem 2.6 and theorem 2.10 if a self-map f_i is homomorphism defined on a commutative ring *R* with unity, and then F_{f_i} is proper ideal of *R*. Again by Theorem 2.7, we can say that, F_{f_i} is proper Nil ideal of *R*.

Remark 2.10: If *R* is a ring with unity say 1, then we can define f_i for *i* = 0 as $f_0(x) = x + 1 \ \forall x \in R$. But, then we have, $f_0(x) \neq x$ for any *x* in *R*. therefore f_0 has no fixed point. Hence, $F_{f_0} = \Phi$.

Theorem 2.10: For a ring $(R, +, \cdot)$, consider a self-map f_i on R given by $f_i(x) = x + x^i$ for each $x \in R$ then F_{f_i} is subset of F_{f_k} if $i \leq k$.

Proof: Consider $x \in F_{f_i}$. Then according to Theorem 2.1, we have, $x^i = 0$. Consider $i \leq k$ so that, $k = i + j, j \geq 0$. Hence, $x^k = x^{(i+j)} = (x^i . x^j) = 0$. Hence by Theorem 2.1, *x* is a fixed point of f_k. Hence, $x \in F_{f_k}$. Thus, F_{f_i} is subset of F_{f_k}. This completes the proof.

Remark 2.11: Using Theorem 2.10, we can conclude that, there exist an ascending chain of subsets F_{f_i}, $i \in N$ as $F_{f_1} \subseteq F_{f_2} \subseteq F_{f_3} \subseteq \cdots\cdots\cdots \subseteq F_{f_k} \subseteq \cdots\cdots\cdots \subset R$.

3. Conclusion

Thus, in this paper we have proved that the set F_{f_i} of fixed points of f_i forms the subring the of *R* under certain conditions defined on f_i. If in particular, *R* is a commutative ring then the set F_{f_i} forms an ideal of *R*, Moreover, it forms a Nil ideal of *R*. Also we have observed that there exist an ascending chain of set of fixed points for different f_i.

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